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ON DEFINING RELATIONS OF CERTAIN INFINITE-DIMENSIONAL LIE ALGEBRAS[1]

BY OFER GABBER AND VICTOR G. KAC

ABSTRACT. In this note we prove a conjecture stated in [2] about defining relations of the so-called Kac-Moody Lie algebras. In the finite-dimensional case this is Serre's theorem [5]. The basic idea is to map the ideal of relations into a Verma module and then to use the (generalized) Casimir operator (cf. [3, 4]).

1. The main statements. Let $A = (a_{ij})$ be an $n \times n$ matrix over a field $\mathbf{F}$. Denote by $\tilde{\mathfrak{g}}(A)$ the Lie algebra over $\mathbf{F}$ with $3n$ generators e_i, f_i, h_i, $i \in I = \{1, \ldots, n\}$ and the following defining relations ($i, j \in I$):

$$(1) \qquad [e_i, f_j] - \delta_{ij}h_i, \quad [h_i, h_j], \quad [h_i, e_j] - a_{ij}e_j, \quad [h_i, f_j] + a_{ij}f_j.$$

Set $\Gamma = \mathbf{Z}^n$, $\Gamma_+ = \{(k_1, \ldots, k_n) \in \Gamma | k_i \geq 0\} \backslash \{0\}$ and let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the standard basis of Γ. Setting $\deg e_i = -\deg f_i = \alpha_i$ for $i \in I$ defines a Γ-gradation $\tilde{\mathfrak{g}}(A) = \bigoplus_{\alpha \in \Gamma} \tilde{\mathfrak{g}}_\alpha$. Let $\tilde{\mathfrak{n}}_\pm = \bigoplus_{\alpha \in \Gamma_+} \tilde{\mathfrak{g}}_{\pm\alpha}$ and $\mathfrak{h} = \tilde{\mathfrak{g}}_0$. Then $\tilde{\mathfrak{n}}_+$ and $\tilde{\mathfrak{n}}_-$ are free Lie algebras over $\mathbf{F}$ with systems of free generators $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$, respectively, and $\tilde{\mathfrak{g}}(A) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$ (direct sum of vector spaces), so that $\tilde{\mathfrak{g}}_{\alpha_i} = \mathbf{F}e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbf{F}f_i$ for $i \in I$, and $\mathfrak{h} = \bigoplus_i \mathbf{F}h_i$ [2, Chapter I]. Define $(\alpha \mapsto \bar{\alpha}) \in \mathrm{Hom}_{\mathbf{Z}}(\Gamma, \mathfrak{h}^*)$ by $\bar{\alpha}_i(h_j) = a_{ji}$ for $i, j \in I$.

Let $\mathfrak{r}$ be the sum of all graded ideals in $\tilde{\mathfrak{g}}(A)$ intersecting $\mathfrak{h}$ trivially. We have the induced gradation $\mathfrak{r} = \bigoplus_{\alpha \in \Gamma} \mathfrak{r}_\alpha$. Setting $\mathfrak{r}_\pm = \mathfrak{r} \cap \tilde{\mathfrak{n}}_\pm$, we obtain that $\mathfrak{r} = \mathfrak{r}_+ \oplus \mathfrak{r}_-$ is a direct sum of ideals.

Our main result is the following.

THEOREM 1. *For $\alpha = (k_1, \ldots, k_n) \in \Gamma$ set*

$$T_\alpha = \sum_{1 \leq i < j \leq n} a_{ij}k_ik_j + \sum_{1 \leq i \leq n} a_{ii}\frac{1}{2}(k_i^2 - k_i)$$

and assume that the matrix A is symmetric. Then the ideal $\mathfrak{r}_+$ (resp. $\mathfrak{r}_-$) is generated as an ideal in $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) by those $\mathfrak{r}_\alpha$ (resp. $\mathfrak{r}_{-\alpha}$) for which $\alpha \in \Gamma_+ \backslash \Pi$ and $T_\alpha = 0$.

COROLLARY [4, THEOREM 1]. *If $T_\alpha \neq 0$ for all $\alpha \in \Gamma_+ \backslash \Pi$, then $\mathfrak{r} = 0$.*

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The next corollary is, in fact, the purpose of the note. An $n \times n$ matrix $A = (a_{ij})$ over a field $\mathbf{F}$ of characteristic 0 is called a *Cartan matrix* iff it satisfies the following properties:

(i) $a_{ii} = 2$, a_{ij} are nonpositive integers for $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$, $i, j \in I$;

(ii) there exists a nondegenerate diagonal $n \times n$ matrix D such that the matrix DA is symmetric.

Define automorphisms s_i, $i \in I$, of the lattice Γ by $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$, $j \in I$; let $W \subset \operatorname{Aut} \Gamma$ denote the group generated by s_i, $i \in I$ [2].

THEOREM 2. *Let* char $\mathbf{F} = 0$ *and let A be a Cartan matrix. Then the elements*

(2) $(\operatorname{ad} e_i)^{-a_{ij}+1} e_j$ *for* $i, j \in I$, $i \neq j$,

(3) $(\operatorname{ad} f_i)^{-a_{ij}+1} f_j$ *for* $i, j \in I$, $i \neq j$,

lie in $\mathfrak{r}$ *and generate the ideals* $\mathfrak{r}_+$ *and* $\mathfrak{r}_-$, *respectively.*

PROOF. It is well known that the property (i) of A implies that all the elements (2) and (3) lie in $\mathfrak{r}$ (see, e.g., [2, Lemma 9]).

In order to prove that these elements generate $\mathfrak{r}_\pm$, note that replacing h_i by $d_i h_i$, $d_i \in \mathbf{F}^*$ and e_i by $d_i^{-1} e_i$ is equivalent to replacing A by the matrix $B = \operatorname{diag}(d_1, \ldots, d_n)A$. Therefore, by the property (ii) of A we can identify the Lie algebras $\tilde{\mathfrak{g}}(A)$ and $\tilde{\mathfrak{g}}(B)$, where $B = (b_{ij})$ is a symmetric matrix; it is also clear that we can choose d_i's so that b_{ii} are positive rational numbers.

Define a symmetric bilinear form $(\ ,\ )$ on Γ by $(\alpha_i, \alpha_j) = b_{ij}$, $i, j \in I$. Then we have $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. Denote by $\mathfrak{g}$ the quotient of $\tilde{\mathfrak{g}}(A)$ by the ideal generated by all elements (2) and (3), let $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$ be the induced gradation and $\bar{\mathfrak{r}}_\pm$ denote the image of $\mathfrak{r}_\pm$ in $\mathfrak{g}$. We have the induced gradation $\bar{\mathfrak{r}}_\pm = \bigoplus_{\alpha \in \Gamma_+} \bar{\mathfrak{r}}_{\pm\alpha}$.

Recall that there exists $\tilde{s}_i \in \operatorname{Aut} \mathfrak{g}$ such that [2, Lemma 10]

$$\tilde{s}_i(\mathfrak{g}_\alpha) = \mathfrak{g}_{s_i(\alpha)} \quad \text{and} \quad \tilde{s}_i(\bar{\mathfrak{r}}_\pm) = \bar{\mathfrak{r}}_\pm.$$

Now suppose that $\bar{\mathfrak{r}}_+ \neq 0$ (the case $\bar{\mathfrak{r}}_-$ is similar). From among $\alpha = (k_1, \ldots, k_n) \in \Gamma_+$ such that $\bar{\mathfrak{r}}_\alpha \neq 0$ choose one of minimal height (i.e., $\Sigma_i k_i$ is minimal). Then height $s_i(\alpha) \geqslant$ height α for all $i \in I$. It follows that $(\alpha, \alpha_i) \leqslant 0$ for all $i \in I$, and hence $(\alpha, \alpha) \leqslant 0$. Hence $2T_\alpha = \Sigma_{i,j} b_{ij}k_i k_j - \Sigma_i b_{ii}k_i < 0$. This is a contradiction with Theorem 1.

COROLLARY 1. *Let* char $\mathbf{F} = 0$ *and let A be an indecomposable Cartan matrix. Let* $\mathfrak{g}(A) = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ *be the Lie algebra with generators* e_i, f_i, h_i, $i \in I$, *and defining relations* (1), (2), (3), *and the gradation induced from* $\tilde{\mathfrak{g}}(A)$. *Set* $\mathfrak{c} = \{h \in \mathfrak{g}_0 = \mathfrak{h} \mid \bar{\alpha}_i(h) = 0$ *for all* $i \in I\}$. *Then*

(a) $\mathfrak{c}$ *is the center of* $\mathfrak{g}(A)$ *and any proper graded ideal of* $\mathfrak{g}(A)$ *lies in* $\mathfrak{c}$.

(b) *Provided that A is not one of the affine matrices from Tables* 1–3 [1], *the Lie algebra* $\mathfrak{g}(A)/\mathfrak{c}$ *is simple.*[2]

PROOF. (a) follows from Theorem 2 and [2, Lemma 1]. (b) follows from (a), [2, Lemma 6], which gives a sufficient condition for nonexistence of a non-graded ideal in $\widetilde{\mathfrak{g}}(A)/\mathfrak{r}$, and [1, §2, Exercise 8b], which implies that this condition holds unless A is affine.

COROLLARY 2. *Let* $A = (a_{ij})$ *be a Cartan matrix and let* $\mathfrak{n}(A)$ *denote the Lie algebra over a field of characteristic* 0 *with generators* $e_1, \ldots, e_n$ *and defining relations* $(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$, $i \neq j$. *Setting* $\deg e_i = \alpha_i$ *defines a* Γ_+*-gradation* $\mathfrak{n}(A) = \bigoplus_\alpha \mathfrak{n}_\alpha$. *For* $w \in W$ *denote by* $s(w)$ *the (finite) sum of the* $\alpha \in \Gamma_+$ *for which* $-w^{-1}(\alpha) \in \Gamma_+$. *Then*

$$\prod_{\alpha \in \Gamma_+} (1 - e^\alpha)^{\dim \mathfrak{n}_\alpha} = \sum_{w \in W} (\det w) e^{s(w)}.$$

PROOF. This follows from Theorem 2 and the "denominator" identity proved in [3]. We remark that the proof in [3] works for the Lie algebra $\widetilde{\mathfrak{g}}(A)/\mathfrak{r}$ (but not $\mathfrak{g}(A)$). Thus the last corollary of [3] (in which Theorem 2 is claimed) remained there unproven.

2. **Proof of Theorem 1.** First, we prove a simple general result on Lie algebras and then apply it to our situation. For a Lie algebra $\mathfrak{p}$ over $\mathbf{F}$, $U(\mathfrak{p})$ will denote its universal enveloping algebra and $U_0(\mathfrak{p}) \subset U(\mathfrak{p})$ the augmentation ideal.

Let $\widetilde{\mathfrak{p}}$ be a Lie algebra over $\mathbf{F}$, $\mathfrak{a}$ an ideal, $\mathfrak{p} = \widetilde{\mathfrak{p}}/\mathfrak{a}$ and $\pi: \widetilde{\mathfrak{p}} \longrightarrow \mathfrak{p}$ the canonical map. The injection $\mathfrak{a} \longrightarrow U_0(\widetilde{\mathfrak{p}})$ and the map π induce homomorphisms of left $\mathfrak{p}$-modules, respectively $\lambda: \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}] \longrightarrow U_0(\widetilde{\mathfrak{p}})/\mathfrak{a}U_0(\widetilde{\mathfrak{p}})$ and $\phi: U_0(\widetilde{\mathfrak{p}})/\mathfrak{a}U_0(\widetilde{\mathfrak{p}}) \longrightarrow U(\mathfrak{p})$, so that $\operatorname{Im} \phi = U_0(\mathfrak{p})$.

LEMMA 1. *The following sequence of* $\mathfrak{p}$*-modules is exact*

$$0 \longrightarrow \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}] \xrightarrow{\lambda} U_0(\widetilde{\mathfrak{p}})/\mathfrak{a}U_0(\widetilde{\mathfrak{p}}) \xrightarrow{\phi} U_0(\mathfrak{p}) \longrightarrow 0. \tag{4}$$

PROOF. The inclusion $\operatorname{Im} \lambda \subset \operatorname{Ker} \phi$ is clear. To show the other inclusion note that $U(\mathfrak{p}) = U(\widetilde{\mathfrak{p}})/\mathfrak{a}U(\widetilde{\mathfrak{p}})$. Hence

$$\operatorname{Ker} \phi = (U_0(\widetilde{\mathfrak{p}}) \cap \mathfrak{a}U(\widetilde{\mathfrak{p}}))/\mathfrak{a}U_0(\widetilde{\mathfrak{p}}) = \mathfrak{a}U(\widetilde{\mathfrak{p}})/\mathfrak{a}U_0(\widetilde{\mathfrak{p}}).$$

As $U(\widetilde{\mathfrak{p}}) = \mathbf{F} \oplus U_0(\widetilde{\mathfrak{p}})$, we see that $\operatorname{Ker} \phi \subset \operatorname{Im} \lambda$.

Finally, we show that $\operatorname{Ker} \lambda = 0$. This is equivalent to $\mathfrak{a} \cap \mathfrak{a}U_0(\widetilde{\mathfrak{p}}) = [\mathfrak{a}, \mathfrak{a}]$. A standard Poincaré-Birkhoff-Witt theorem argument gives that this is equivalent to $\mathfrak{a} \cap U_0(\mathfrak{a})^2 = [\mathfrak{a}, \mathfrak{a}]$ (in $U(\mathfrak{a})$). The inclusion $\supset$ is obvious, and the other inclusion is shown by passage to $U(\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}])$.

[2] For an affine matrix A there is an explicit construction of $\mathfrak{g}(A)/\mathfrak{c}$ [2], which shows that (b) fails in this case.

Recall that if V is a left module over a Lie algebra $\mathfrak{a}$, then given a Lie algebra homomorphism $\psi: \mathfrak{a} \longrightarrow \mathfrak{b}$, one defines the *induced* left $\mathfrak{b}$-module

$$\operatorname{Ind}_{\mathfrak{a}}^{\mathfrak{b}} V = (U(\mathfrak{b}) \otimes_{\mathbf{F}} V)/\Sigma_{a,b,v}\mathbf{F}\,(b\psi(a) \otimes v - b \otimes a \cdot v),$$

where $b \in U(\mathfrak{b})$, $a \in \mathfrak{a}$, $v \in V$, with an obvious action of $\mathfrak{b}$.

Now we turn to the Lie algebra $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}(A)$ associated to a symmetric matrix $A = (a_{ij})$. Set $\mathfrak{g}' = \widetilde{\mathfrak{g}}/\mathfrak{r}$; denote by π the canonical homomorphism $\widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}'$. We have the induced gradation $\mathfrak{g}' = \bigoplus_{\alpha\in\Gamma} \mathfrak{g}'_\alpha$ and induced decomposition $\mathfrak{g}' = \mathfrak{n}'_- \oplus \mathfrak{h} \oplus \mathfrak{n}'_+$ (we identify $\mathfrak{h}$ with $\pi(\mathfrak{h}) = \mathfrak{g}'_0$), so that $\mathfrak{g}'_{-\alpha_i} = \mathbf{F}\pi(f_i)$, $\mathfrak{g}'_{\alpha_i} = \mathbf{F}\pi(e_i)$ for $i \in I$. Define a symmetric bilinear form on Γ by $(\alpha_i, \alpha_j) = a_{ij}$ for $i, j \in I$.

For $\alpha \in \Gamma$ define a $\widetilde{\mathfrak{g}}$-module $\widetilde{M}(\alpha) = \operatorname{Ind}_{\mathfrak{h}\oplus\widetilde{\mathfrak{n}}_+}^{\widetilde{\mathfrak{g}}} \mathbf{F}_\alpha$, where $\mathbf{F}_\alpha$ is a 1-dimensional module with underlying space $\mathbf{F}$ defined by $\widetilde{\mathfrak{n}}_+(1) = 0$, $h(1) = \bar{\alpha}(h)$ for $h \in \mathfrak{h}$. Denote the image of $1 \otimes 1$ in $\widetilde{M}(\alpha)$ by $\widetilde{v}_\alpha$ and let $\widetilde{M}(\alpha)_\alpha = \mathbf{F}\cdot\widetilde{v}_\alpha$. The Γ-gradation of $\widetilde{\mathfrak{g}}$ induces a gradation $\widetilde{M}(\alpha) = \bigoplus_{\eta\in\Gamma_+\cup\{0\}}\widetilde{M}(\alpha)_{\alpha-\eta}$, so that $\widetilde{\mathfrak{g}}_\beta\widetilde{M}(\alpha)_\gamma \subset \widetilde{M}(\alpha)_{\beta+\gamma}$. Further on, by a "module" we mean a Γ-graded module. $\widetilde{M}(\alpha)$ contains a unique proper maximal submodule which is denoted by $\widetilde{M}^1(\alpha)$. Similarly, we define the $\mathfrak{g}'$-module $M(\alpha) = \operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}'_+}^{\mathfrak{g}'} \mathbf{F}_\alpha$, its gradation $M(\alpha) = \bigoplus_\eta M(\alpha)_{\alpha-\eta}$, the canonical generator v_α, the submodule $M^1(\alpha)$, etc.

An element $v \in M_\gamma$ in a Γ-graded $\mathfrak{g}'$-module M is called *primitive of weight* γ iff there exists a $\mathfrak{g}'$-submodule $V \subset M(\alpha)$ such that $v \notin V$ but $\mathfrak{n}'_+(v) = 0 \bmod V$; then we call $\gamma \in \Gamma$ a *primitive weight* of M.

LEMMA 2. *If $\alpha - \beta$ is a primitive weight for the $\mathfrak{g}'$-module $M(\alpha)$, then $T_\beta = (\alpha, \beta)$.*

PROOF. Define the (generalized) Casimir operator Ω on $M(\alpha)$ by

$$\Omega(v) = (T_\eta - (\alpha, \eta))v + \sum_{\gamma\in\Gamma_+} \sum_i e_{-\gamma}^{(i)} e_\gamma^{(i)}(v) \quad \text{if } v \in M(\alpha)_{\alpha-\eta},$$

where $e_\gamma^{(i)}$ is a basis of $\mathfrak{g}'_\gamma$ and $e_{-\gamma}^{(i)}$ is a dual basis of $\mathfrak{g}'_{-\gamma}$ with respect to the invariant symmetric bilinear form on $\mathfrak{g}'$ as in [4, p. 313]. This has been introduced in a slightly different form in [3] and differs from the version in [4] only by a factor ½ (we do it in order to include the case char $\mathbf{F} = 2$). As usual, one shows by a direct computation that Ω commutes with e_i and f_i action, and as $\Omega(v_\alpha) = 0$, obtains that $\Omega = 0$. On the other hand, by the definition of Ω, a primitive vector $v \in M(\alpha)_{\alpha-\beta}$ is an Ω-eigenvector modulo a submodule, with eigenvalue $T_\beta - (\alpha, \beta)$. Hence, this eigenvalue is 0.

Now we are able to complete the proof of Theorem 1. We apply the exact sequence (4) to $\widetilde{\mathfrak{p}} = \widetilde{\mathfrak{n}}_-$ and $\mathfrak{p} = \mathfrak{n}'_-$. We clearly have the following isomorphisms of $\widetilde{\mathfrak{n}}_-$-modules: $U_0(\widetilde{\mathfrak{n}}_-) = \widetilde{M}^1(0) = \bigoplus_{i=1}^n \widetilde{M}(-\alpha_i)$; the last isomorphism (of $\widetilde{\mathfrak{g}}$-modules, actually) is due to the fact that $\widetilde{\mathfrak{n}}_-$ is a free Lie algebra and hence

$U(\tilde{\mathfrak{n}}_-)$ is freely generated by f_i, $i \in I$. We also have the following isomorphisms of $\mathfrak{n}'_-$-modules: $U_0(\mathfrak{n}'_-) = M^1(0)$ and $U_0(\tilde{\mathfrak{n}}_-)/\mathfrak{r}_-U_0(\tilde{\mathfrak{n}}_-) = U(\mathfrak{n}'_-) \otimes_{U(\tilde{\mathfrak{n}}_-)} U_0(\tilde{\mathfrak{n}}_-) = U(\mathfrak{n}'_-) \otimes_{U(\tilde{\mathfrak{n}}_-)} \tilde{M}^1(0) = U(\mathfrak{n}'_-) \otimes_{U(\tilde{\mathfrak{n}}_-)}(\bigoplus_{i=1}^n \tilde{M}(-\alpha_i)) = \bigoplus_{i=1}^n M(-\alpha_i)$. Hence (4) gives an exact sequence of $\mathfrak{n}'_-$-modules,

$$0 \longrightarrow \mathfrak{r}_-/[\mathfrak{r}_-, \mathfrak{r}_-] \xrightarrow{\lambda} \bigoplus_{i=1}^n M(-\alpha_i) \xrightarrow{\phi} M^1(0) \longrightarrow 0. \tag{5}$$

Now we show that (5) is, in fact, an exact sequence of $\mathfrak{g}'$-modules. For the map ϕ this is clear. To show that λ is a $\mathfrak{g}'$-module homomorphism we describe it more explicitly. Define $\psi: \mathfrak{r} \longrightarrow \tilde{M}^1(0)$ by $\psi(a) = a(\tilde{v}_0)$. This induces $\lambda_1: \mathfrak{r} \longrightarrow U(\mathfrak{g}') \otimes_{U(\tilde{\mathfrak{g}})} \tilde{M}^1(0)$ such that $\lambda_1(\mathfrak{r}_+) = 0, \lambda_1([\mathfrak{r}_-, \mathfrak{r}_-]) = 0$, which gives us the map λ. We have to check that λ_1 is a homomorphism of $\tilde{\mathfrak{g}}$-modules. Indeed, for $a \in \mathfrak{r}$ and $x \in \tilde{\mathfrak{g}}$ one has

$$\begin{aligned} \lambda_1([x, a]) &= 1 \otimes (xa\tilde{v}_0 - ax\tilde{v}_0) = \pi(x) \otimes a\tilde{v}_0 - \pi(a) \otimes x\tilde{v}_0 \\ &= \pi(x) \otimes a\tilde{v}_0 = \pi(x)\lambda_1(a). \end{aligned}$$

Now let $-\alpha$ be a primitive weight of the $\mathfrak{g}'$-module $\mathfrak{r}_-/[\mathfrak{r}_-, \mathfrak{r}_-]$. Then, since (5) is an exact sequence of $\mathfrak{g}'$-modules, we deduce that $-\alpha$ is also a primitive weight of one of the $\mathfrak{g}'$-modules $M(-\alpha_i)$ and hence of the $\mathfrak{g}'$-module $M(0)$. Hence, by Lemma 2, we obtain that $T_\alpha = 0$. $\alpha \notin \Pi$ since no f_i lies in $\mathfrak{r}$. As the $\mathfrak{n}'$-module $\mathfrak{r}_-/[\mathfrak{r}_-, \mathfrak{r}_-]$ is generated by primitive vectors (because a homogeneous vector v is not primitive iff $v \in U(\mathfrak{n}'_-)U(\mathfrak{n}'_+)\mathfrak{n}'_+ \cdot v$) we obtain that the ideal $\mathfrak{r}_-$ in $\tilde{\mathfrak{n}}_-$ is generated by those $\mathfrak{r}_{-\alpha}$ for which $\alpha \in \Gamma_+\backslash\Pi$ and $T_\alpha = 0$, as required. The result for $\mathfrak{r}_+$ follows by applying the involution θ of $\tilde{\mathfrak{g}}$ defined by $\theta(e_i) = -f_i$, $\theta(f_i) = -e_i$, $\theta(h_i) = -h_i$, $i \in I$.

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DEPARTMENT OF MATHEMATICS, TEL AVIV UNIVERSITY, RAMAT AVIV, ISRAEL

DEPARTMENT OF MATHEMATICS, MASSACHUSETS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139