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BOOK REVIEWS

Permutation groups and combinatorial structures, by N. L. Biggs and A. T. White, London Mathematical Society Lecture Note Series No. 33, Cambridge Univ. Press, New York and London, 1979, viii + 140 pp., $13.95.

The close relationships between group theory and structural combinatorics go back well over a century. Given a combinatorial object, it is natural to consider its automorphism group. Conversely, given a group, there may be a nice object upon which it acts. If the group is given as a group of permutations of some set, it is natural to try to regard the elements of that set as the points of some structure which can be at least partially visualized.

For example, in 1861 Mathieu **[8]**, **[9]** discovered five multiply transitive permutation groups. These were constructed as groups of permutations of 11, 12, 22, 23 or 24 points, by means of detailed calculations. In a little-known 1931 paper of Carmichael **[5]**, they were first observed to be automorphism groups of exquisite finite geometries. This fact was rediscovered soon afterwards by Witt **[11]**, who provided direct constructions for the groups and then the geometries. It is now more customary to construct first the designs, and then the groups, using projective planes (as in Lüneburg **[7]**) or codes (as in Cameron **[2]**, or Cameron and van Lint **[3]**). This change of point of view should be compared with the corresponding phenomenon in the case of the classical groups: they are now regarded as groups of linear transformations acting on a vector space, rather than as groups of matrices to be laboriously multiplied, inverted and conjugated.

In order to see more precisely how groups and combinatorial objects are related, consider a finite group G acting transitively on a set X; the elements of X will be called "points". Assume for the moment that G is 2-transitive. Take any subset B of X such that $2 \leqslant |B| < |X|$, and form all the distinct images of B by the elements of G. Any two images have the same size. Since any pair of points can be moved to any other pair by a suitable element of G, the number of images containing a pair is constant. Thus, this is an example of a *design* (or "balanced incomplete block design" in the statistical literature): X is its set of points, and the images of B are its "blocks". Designs play an important role in combinatorics, even when no group is present.

Of course, the above construction is much too general to give useful information about the group: B was chosen arbitrarily, and need not have much to do with the way G acts on X. Careful choices must be made. For example, if G is the collineation group of a finite projective space, the most natural choices for B are the subspaces of that space. The hindsight provided by the determination of all finite simple groups tells us that, if G is any 2-transitive group other than the alternating or symmetric group on X, then very good choices exist for B. However, there is no uniform choice for B which is guaranteed to reflect interesting properties of G.

If G is transitive but not 2-transitive on X, there is a more natural combinatorial object arising from G than in the 2-transitive case. Let R be any orbit of G on $X \times X$ other than $\{(x, x) | x \in X\}$. Then R is a relation on X, and defines a directed graph. Clearly, G acts transitively on both the vertices and edges of this directed graph. There are at least two choices for the orbit R, since G is not 2-transitive on X. Moreover, G is primitive on X if and only if each such directed graph is connected. However, once again this construction is too general: each proper subgroup H of a given group G produces a transitive permutation representation of G, which is primitive if and only if H is a maximal subgroup. Many groups (such as those of prime power order) do not appear to have any interesting permutation representations.

Thus, the study of a permutation group by means of designs or graphs requires additional hypotheses. Nevertheless, there have been many situations in which such hypotheses have arisen naturally, and the resulting combinatorial outlook has been a valuable asset in the study or characterization of especially important types of permutation groups.

The combinatorial point of view has also been important for construction of new and interesting groups – especially of simple ones. One type of construction proceeds as follows. Let H be a permutation group on X, fixing some $x \in X$. Assume that H has been chosen carefully, and that a great deal is known about the action of H on X. Conceivably, there is a *transitive extension* of H: a group G acting transitively on X such that H is the stabilizer of x in G. If H was transitive on $X - \{x\}$, then G will be 2-transitive on X. If H was intransitive on $X - \{x\}$, then G will have to act on directed graphs obtained as above. In especially nice (i.e., very carefully chosen) situations, a highly structured combinatorial object can be found using H and X, having X as its set of points and H as a group of automorphisms, such that G has very small index in the group of all automorphisms. For example, in one standard construction for the Mathieu group M_{22}, the set $X - \{x\}$ consists of the 21 points of the projective plane $PG(2, 4)$ over $GF(4)$, while H is the group of collineations arising from 3×3 matrices of determinant 1. Similarly the Higman-Sims simple group arises with $H = M_{22}$ and $X - \{x\}$ the union of the sets of 22 points and 77 blocks of the M_{22}-design. These examples produce a design and a graph, respectively; in each case, the desired simple group has index 2 in the full automorphism group.

The relationships between permutation groups and combinatorics are far from one-sided. The most important, highly structured combinatorial objects tend to have large automorphism groups. Thus, the study of structural combinatorics naturally leads to group theoretic situations.

The book under review presents a brief introduction to structural combinatorics and permutation groups. This is the fourth book in the London Mathematical Society Lecture Note Series which deals with these subjects at approximately the level of a first or second year graduate student. The other three are by Cameron [**2**], Cameron and van Lint [**3**], and Biggs [**1**] (the present book being intended as a replacement for the last of these).

Biggs and White have as their major goal the construction of the Mathieu and Higman-Sims simple groups and their associated designs and graph. Far from heading directly towards these topics, they carefully build up the numerous group theoretic and combinatorial preliminaries. The authors' research has dealt primarily with the combinatorial side of the subject matter. This has resulted in extremely slow and meticulous handling of groups; but it has not prevented the choice of topics from being excellent, whether the reader's point of view is combinatorial or algebraic.

Chapter 1 presents a general discussion of permutation groups, especially multiply transitive or at least primitive ones. It is similar to earlier parts of Wielandt [**10**], but more informal. The simplicity of the alternating groups A_n, $n \geqslant 5$, is handled nicely. Chapter 2 is concerned with finite geometries. The 7 point projective plane is drawn, and its group is defined and shown to be simple. The general and special linear groups are defined, and are seen to act on projective spaces. The usual simplicity result is proved very slowly, using transvections. This links up nicely with the multiply transitive groups portion of Chapter 1. Symplectic, orthogonal and unitary groups are briefly defined, and simplicity is mentioned.

Chapter 3 concerns designs. It begins with elementary counting and incidence matrix results, followed by a proof of the Bruck-Chowla-Ryser theorem; few of these general results reappear later. Designs are constructed from multiply transitive groups as indicated above. Permutations and calculations are provided in order to obtain the existence of the Mathieu group M_{22}; a sketch is given for M_{23} and M_{24}. (The groups M_{11} and M_{12} are assumed to be known from a very long sequence of highly computational exercises in Chapter 1.) The groups M_{22}, M_{23} and M_{24} are proved to be simple; the simplicity of M_{12} is a starred (i.e., difficult) exercise at the end of another sequence of fairly difficult exercises. The beautiful design associated with M_{22} is now constructed, and the existence of corresponding designs for the remaining Mathieu groups is mentioned. Witt's uniqueness result for the 22 point design is left to another sequence of exercises.

It is unfortunate that the existence of the Mathieu groups is obtained in such an old-fashioned, computational manner. Of course, computation has the advantage of requiring relatively little space if it is mostly left to the reader. However, it also masks most of the beautiful properties of the associated designs while involving ideas which the reader is unlikely to need again. Fortunately, elegant constructions exist which exhibit many remarkable relationships among all five designs and the 21 point projective plane $PG(2, 4)$ (Cameron [**2**]; Lüneburg [**7**]).

Chapter 4 considers transitive groups which are not 2-transitive. Directed graphs are obtained as above. The diameter 2 case (strongly regular graphs) is studied in some detail using eigenvalues of adjacency matrices. The culmination of the first four chapters is the construction of the Higman-Sims graph and its associated sporadic simple group HS. This construction requires familiarity with permutation groups, projective planes, designs and graphs. It is a fitting way to end the main portion of the book. The final chapter concerns automorphism groups of maps on surfaces. Frobenius groups and

Paley maps are discussed; genus is defined and then computed for symmetrical maps.

The book is unusually error-free. The only serious mistake occurs in the discussion of HS. If A and B are permutation groups on X, and $0 \in X$, then $\langle A, B\rangle_0$ usually does not coincide with $\langle A_0, B_0\rangle$. This invalidates the authors' approach to the simplicity of HS. Since the full automorphism group of the 22 point M_{22}-design was never discussed, their determination of the order of HS is also incomplete. The reader should refer instead to the paper by Higman and Sims [6] or to Lüneburg [7].

The presentation involves a very pleasant blend of informal discussions and formal proofs. The prerequisites are few: some linear algebra and an undergraduate-level familiarity with groups. Nevertheless, the book comes reasonably close to recent research, and conveys this fact clearly.

The book is not quite a text. Exercises mostly come in long sequences aimed at a single result, and are frequently difficult. For example, the uniqueness of the 22 point M_{22}-design constitutes a page long sequence of exercises, a complete mastery of which is, moreover, essential for the entire discussion of HS. Nearfields, the 3-dimensional unitary group over $GF(3)$, Hadamard matrices, and examples of graphs and designs appear in other sequences. One exercise (5.7.6) is a short, starred one, consisting of an unreferenced result of Carlitz [4], whose complicated proof involves none of the concepts appearing in this book.

Since the book is in a lecture note series, it is perhaps understandable that few references are given. However, as part of the informal manner of presentation, many informative remarks are made without any references to proofs and little indication of what is easy to prove or what is at the research level. Each chapter ends with a very short bibliography. Most assertions can be found somewhere within some reference; finding the right page of the correct book would not be easy for a student. Fortunately, these assertions do not contain errors. (The one unfortunate remark involves what might be regarded as a matter of opinion. The statement (p. 49) that the occurrence of sporadic simple groups "accounts for much of the continuing interest in finite groups" is not correct.)

An instructor familiar with this area would be able to indicate references, point out what is easy and what is difficult, and hence transform the authors' remarks into large numbers of exercises. Such guidance would turn this into a very useful text. The classification of all finite simple groups will have numerous consequences (other than instantaneous ones) for both permutation groups and combinatorics – and, undoubtedly, elsewhere in mathematics. This prospect should further increase the value of books such as this one.

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Vector bundles on complex projective spaces, Progress in Mathematics, vol. 3, by Christian Okonek, Michael Schneider and Heinz Spindler, Birkhäuser, Boston, 1980, vii + 389 pp., $18.00 paperback.

This book is an introduction to the basic theory and classification of holomorphic vector bundles on complex projective spaces. A holomorphic vector bundle on a complex manifold M is just what you would expect: It is a locally trivial fibre space E over M with fibre $\mathbf{C}^r$ and with transition functions which are holomorphic on the base. The dimension r of the fibre is called the *rank* of the bundle.

Smooth vector bundles (that is, with C^∞ transition functions) have been known for a long time in differential geometry. They can in principle be classified by certain characteristic classes (in the case of $\mathbf{C}^r$-bundles their Chern classes) and some homotopy invariants. The study of holomorphic vector bundles is more recent, and the classification problem is of an essentially different nature because of the extra structure imposed by holomorphic functions. Once the topological type of the underlying smooth vector bundle has been fixed, one finds in general continuous families of nonisomorphic holomorphic bundles. The parameter spaces of these families are called *moduli spaces*. The classification problem thus consists of determining which smooth $\mathbf{C}^r$-bundles carry a holomorphic structure, and then describing the moduli space of the possible holomorphic bundles.

Holomorphic vector bundles on compact Riemann surfaces were studied extensively beginning in the 1960's. Then attention turned to higher-dimensional manifolds and in particular, there has been a big spurt of recent activity concerning holomorphic vector bundles on complex projective spaces, the subject of this book. The extent of this activity can be judged from the bibliography of this volume, which contains 138 items, about half of which date since 1977.

Why is this subject suddenly so popular? I see three principal reasons. One