complaint is that the authors always use the language of complex manifolds and holomorphic sheaves, and do not point out that everything they say makes sense in abstract algebraic geometry over an algebraically closed groundfield k. In fact, with the exception of Chapter II §2 on the generic splitting type of a semistable bundle, there is no need even to suppose the groundfield of characteristic 0.

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Elements of soliton theory, by G. L. Lamb, Jr., Wiley, New York, 1980, xii + 289 pp., $29.95.

Until comparatively recently in the history of science, mathematics and physics lived in close relation. The advance of physics suggested the development of new mathematics, and conversely the progress of mathematics fed into the way physicists thought about nature. This contact remained strong until the 1920's, when it began to ebb, reaching a low in the 1950's. What is beginning to bring us together again are two striking developments of science in the 1960's–the successful application of Lie group theory and differential geometry to elementary particle physics and the theory of *solitons*.

The first topic was the result of efforts by a thundering herd. In contrast to this 'big science', the theory of solitons was the remarkable creation of a small group–Kruskal, Zabusky, Gardner, Greene and Miura–working at Princeton (but not in the mathematics or physics department!). That the roots of the theory lie in the 19th century work on hydrodynamics of Scott Russell and computer studies of solutions of nonlinear differential equations by Fermi, Pasta and Ulam in the 1950's has been recounted many times, [**1**, **2**], and will not be repeated here.

The Princeton group produced a complex of original ideas that has had a remarkable success and influence in contemporary science. Their theory was developed with the traditional tools of the physically-minded applied mathematician: classical differential equation theory, the mathematics of quantum mechanics, scattering theory, etc. A remarkable quality of their work is that it linked 19th century analysis and geometry with some of the most modern parts of functional analysis, differential, and algebraic geometry.

This book is an admirable attempt by a physicist-applied mathematician to present an introductory account of the theory of solitons from some intermediate point on the scale of mathematical sophistication. Lamb explicitly excludes an attempt to describe the geometric or Lie theoretic side of the theory, although his own research work is in this direction. Since a mathematician who comes to the subject cold and tries to read this book or the original literature [**3**, **4**] will probably have difficulty seeing the forest for the trees, and the 'geometric' point of view is valuable precisely for the overall perspective it provides, I will devote this review to a description of certain qualitative insights into soliton theory that I have found most useful in my own education in this area.

Let us begin with the original work by Kruskal and Zabusky, who coined the term 'soliton' to denote certain particle-like objects which occur in nonlinear field theories. Consider a partial differential equation in two independent variables, that we label 'x' and 't', and one dependent variable, labelled 'u',

$$f(u, \partial u/\partial t, \partial u/\partial x, \dots) = 0. \tag{1}$$

Introduce the space $J^\infty(R^2, R)$ of *infinite order jets* of maps of $R^2 \to R$. In this case, $J^\infty(R^2, R)$ is the Cartesian product of a countable number of copies of R. Label a point j of $J^\infty(R^2, R)$ as follows,

$$j = (t, x, u, u_x, u_t, u_{xx}, \dots).$$

Given a C^∞ map: $R^2 \to R$,

$$(t, x) \to u(x, t),$$

introduce its *prolongation* $j^\infty(u)$ as a map $R^2 \to J^\infty(R^2, R)$ defined as follows:

$$j^\infty(u)(t, x) = (t, x, u(x, t), (\partial u/\partial x)(x, t), \dots).$$

Let **F** be the algebra (under pointwise multiplication) of real-valued functions on $J^\infty(R^2, R)$ which depend on only a finite number of coordinates, and are C^∞ functions of these coordinates. Thus, the function f on the left-hand side of (1) is an element of **F**. A *solution* of (1) is a mapping u: $R^2 \to R$ such that

$$f j^\infty(u) = 0.$$

This suggests giving a 'geometric' structure to the space $J^\infty(R^2, R)$ based on the *algebraic* properties of **F**, e.g., defining *vector fields* as derivations of **F** and *differential forms* as dual objects to vector fields. Of most importance in the soliton work, are the differential forms of the following type:

$$dx, dt, du \tag{2}$$

$$du - u_x dx - u_t dt, \qquad du_t - u_{tx} dx - u_{tt} dt \tag{3}$$

and so on,

$$\omega = f_1 dt + f_2 dx. \tag{4}$$

The forms (2) are the *coordinate differential forms*, the forms (3) the *contact* differential forms (they are zero when pulled back under the prolongation maps) and (4) are the forms (which so far have no standard name) which enter into the concept of 'conservation law'.

DEFINITION. A differential form ω of type (4) is a *conservation law* for the differential equation (1) if its exterior derivative $d\omega$ lies in the differential ideal ('ideal' in the sense of the Grassmann algebra) generated by the 1-forms df and the contact forms (3).

Now, such a conservation law differential form defines a real-valued function h_ω on the space **S** of solutions of the differential equation (1). For each map u: $(x, t) \to u(x, t)$, set

$$h_\omega(u) = \text{integral of the 1-form } \omega \text{ on the line } -\infty < x < \infty, \text{ with } t \text{ held fixed.}$$

(The 'conservation law' condition in fact guarantees that the integral is independent of t.)

Assume a certain ordered set $(\omega_1, \ldots, \omega_m, \ldots)$ of conservation laws is known. The *soliton solutions* of (1) corresponding to this set are obtained by minimizing some of the functions h_ω on **S**, while holding the others constant.

One-solitons. Solutions of (1) which minimize h_{ω_1},

Two-solitons. Solutions of (1) which minimize h_{ω_2} on the subset of **S** where h_{ω_1} is constant. And so on.

In this way, one obtains families of particular solutions of the partial differential equations (1). The solitons are solutions of one-variable calculus of variations problems, hence satisfy ordinary differential equations. Thus at each level the solitons depend on a finite number of parameters. This association with minimization problems is one possible mathematical way of formulating the extraordinary stability properties that first brought them to the attention of physicists. (From the physical point of view, solitons are 'solitary wave' solutions of nonlinear wave equations that preserve their shape under interaction. Of course, it is not known in general how close the connection is between the desired physical properties and the definition via extremal-conservation law properties. There is another possible way of defining solitons, in terms of Baecklund transformations, which seems possibly more closely related to the interaction properties.)

Of course, this method of defining solitons is dependent on an initial choice of conservation laws. In fact, it is very difficult to find such conservation laws. (Differential equations which admit certain types of explicitly-given families of conservation laws are sometimes said to be 'completely integrable', but this is an unfortunate choice of terminology, which conflicts with standard and traditional usage in differential geometry.) Hence after the initial work of Kruskal and Zabusky, which involved this 'brute force' approach for the Korteweg-de Vries equation, attention shifted to studying the underlying geometric and analytic structure which might generate this structure. There seems to be at least three such structures: the 'symplectic structure', the 'inverse scattering structure', and the 'Baecklund structure'. Lamb does not pursue the symplectic approach in this book. (This is a wise decision for an introductory treatise.) Instead he follows the second major approach to soliton theory poineered by the Princeton group in the 1960's, the *isospectral deformatioin-inverse scattering* approach. To explain what is involved here, let us begin with a geometric setting in the finite-dimensional case.

Let $M(n)$ be the vector space of $n \times n$ complex matrices. An *isospectral flow* is a one-parameter group of matrices $t \to g(t)$ in $M(n)$ such that the eigenvalues of matrices (considered as functions on $M(n)$) are invariants of the action. Now, for differential-geometric purposes, the eigenvalues are not ideal choices for functions (since they are not C^∞), and it is better to replace them with their algebraic equivalent, the traces of powers of matrices (or the coefficients of the characteristic polynomial).

Define a map $\pi: M(n) \to C^n$,

$$\pi(A) = (\text{trace}(A), \ldots, \text{trace}(A^n)) \quad \text{for } A \in M(n).$$

Thus, an isospectral flow is a one-parameter group whose orbits lie in the fibers of π.

Let V be the vector field on $M(n)$ which is the infinitesimal generator of the group $t \to g(t)$. Since $M(n)$ is a vector space, V can be regarded simply as a map $M(n) \to M(n)$. (Of course, for general manifolds, a 'vector field' is a cross-section of the tangent bundle.)

Now, there is a Lie group acting on $M(n)$, the group $GL(n, C)$ of invertible $n \times n$ complex matrices,

$$(g, A) \to gAg^{-1}.$$

The orbits of this group action are contained in the fibers of π (classically, the eigenvalues are invariant under matrix similarity). Let$M(n)_0$ be the points of $M(n)$ at which the orbits of $GL(n, C)$ are principal, i.e. are of maximal dimension and, among those of maximal dimension, the isotropy subgroups have a minimal number of components. $M(n)_0$ is the set of matrices with distinct eigenvalues, and is open and dense in $M(n)$. Further, on $M(n)_0$ the orbits of $GL(n, C)$ are equal to the fibers of π, and π restricted to $M(n)_0$ is a submersion. The vector field V which generates an isospectral flow is, when restricted to $M(n)_0$, of the following form

$$V(A) = [B(A), A],$$

where B is a map $M(n)_0 \to M(n)$.

Then the orbit curves $t \to A(t)$ of the isospectral flow (and which lie in $M(n)_0$) are solutions of the following nonlinear ordinary differential equations

$$dA/dt = [B(A), A]. \tag{5}$$

This equation (which had appeared in another physical context in earlier work by Arnold) is called the *Lax equation* by the workers in soliton theory. It is the starting point for almost all of the research of the last ten years. In fact, the usual procedure is to turn the argument around and to define isospectral flows as those generated by differential equations of type (5).

Now, this Lie-theoretic model also leads to a cleaner formulation of what is meant by a 'soliton'. Let M be a manifold, let V be a vector field in M, and let $f_1, f_2, \ldots$ be a set of C^∞ functions on M, each of whose Lie derivative with respect to V is zero. Let N be a subset of M such that the one-parameter group generated by V (whose orbits are the trajectories of V) leaves N invariant. This V generates a 'dynamical system' in N, that is denoted by (V, M, N), and the $f_1, f_2, \ldots$ restricted to N are invariant under the flow.

DEFINITION. The 1-*soliton* orbits of the dynamical system (V, M, N) are the curves $t \to \sigma(t)$ in N such that

$$\sigma(t) \in N \quad \text{for all } t \tag{6}$$

$$d\sigma/dt = V(\sigma(t)) \tag{7}$$

$$(8) \qquad f_1(\sigma(t)) \leq f_1(p) \quad \text{for all } p \in N, \text{ all } t \in R.$$

The 2-*solitons* are the curves $t \to \sigma(t)$ which satisfy (6), (7) and the following conditions,

$$(9) \qquad (d/dt)t_1(\sigma(t)) = 0,$$

$$(10) \qquad f_2(\sigma(t)) \leq f_2(p) \quad \text{for all } p \in N, \text{ all } t \in R.$$

The n-solitons are defined similarly, i.e. as special 'extremal' orbit curves of the flow restricted to N.

The most exciting point (to a Lie theorist) is that this work suggests a setting of an 'infinite-dimensional Lie Theory' of a sort that is yet to be created. In fact, one can look at Lamb's book as providing the raw material (in terms of classical analysis and mathematical physics) for such an ultimate general theory in the case that (M, N) is replaced by the following infinite-dimensional object:

M = space of differential operators of the form

$$a(x)d^2/dx^2 + b(x)d/dx + c(x),$$

N = set of Sturm-Liouville operators

$$(14) \qquad d^2/dx^2 + u(x),$$

$$(15) \qquad f_j(A) = \text{trace}(A^j) \quad \text{for } A \in N.$$

Now, purists might object that the trace of a differential operator is meaningless. In fact, it can be given a meaning, as a result of the marvelous work of Gelfand, Levitan and Dikiĭ in the 1950's [5]. (This work was one of the main underpinnings of the soliton work. The next time your local physicist asks what pure mathematics has done for him lately, you can point to this.)

Now, the famous Korteweg-de Vries equation is the following partial differential equation

$$\partial u/\partial t + u\partial u/\partial x + \partial^3 u/\partial x^3 = 0.$$

A solution is a function $(x, t) \to u(x, t)$ on R^2. Lie-theoretically, it is thought of as the equation (with given Cauchy data) of the orbit of a one-parameter group acting on the vector space U of functions $x \to u(x)$ in R. The corresponding vector field on this vector space is the following third-order, nonlinear differential operation,

$$V(u) = -udu/dx - d^3u/dx^3.$$

Now, one can map U into the set of Sturm-Liouville operators as follows,

$$u \to A_n = d^2/dx^2 + u(x).$$

Then, one readily calculates that

$$A_{V(u)} = [B_u, A_u]$$

where B_u is a third order linear differential operator whose specific form is not important. Thus, by the mapping $u \to A_u$ we shift the 'group' generated by the Korteweg-de Vries equation to a Lie-theoretic setting of the type described earlier, with the group $GL(n, C)$ replaced by an infinite-dimensional monstrosity: the 'group' whose 'Lie Algebra' is the set of all linear ordinary differential operators with C^∞ coefficients. But now comes the miracle: Gelfand and Levitan had shown in 1951 how one could map the space of all Sturm-Liouville operators (on the interval $0 \leq x < \infty$) to a vector space (the 'spectral data') in such a way that the inverse map existed (for a suitably chosen class of u's), and the corresponding u was derived from a solution of a *linear Fredholm integral equation*, determined by the spectral data. For the K-dV work, a modification is required to cover the interval $-\infty < x < \infty$, supplied by work of Kay and Moses.

Thus, as a result of two transformations–the assignment of a Sturm-Liouville operator to a function u of x and the spectral data to the operator–the vector field on the space of u's which determines the K-dV equation is transformed to a linear vector field. This is the famous Inverse Scattering Technique [**4**]. It turns out that soliton solutions of K-dV (of course defined via the original Kruskal-Zabusky conservation law method) go over under this transform to an especially simple and obvious choice of spectral data.

The next interesting point is the physical meaning of the 'spectral data' that goes into the Inverse Scattering equation. Now, the original Gelfand-Levitan work on the interval $0 < x < \infty$ was motivated by quantum mechanics, where the operator is the radial part of a 3-dimensional, spherically-symmetric Schrödinger operator. It turns out that the physical problem which leads most naturally to the Marchenko equation is the propagation of a 'classical' wave of frequency k in a 1-dimensional medium, e.g. a cable. (In fact, this was a major topic in 1930's and 40's electrical engineering.) Part of the wave is 'reflected', part 'transmitted': The precise proportion depends on k. Thus, one obtains two functions, $R(k)$ and $T(k)$, the *reflection* and *transmission* coefficients. These functions of the real variable k are the basic data to be fed into the Inverse Scattering Integral equation.

Turning now to Lamb's book, the core is a uniquely thorough treatment of the complex of ideas around Inverse Scattering from the point of view of a mathematically literate but down-to-earth physicist. I found this material very enjoyable, and believe the mathematical world will find this book of great value as a source for material that only exists as 'folklore' in engineering and physics circles. Of course, it may not be to the taste of someone who likes to see mathematical physics come in pedantically precise functional analysis packages. The 'correct' setting is probably not Hilbert space theory, but some version of the theory of Fourier integral-pseudodifferential operators. In fact, Lamb presents (on p. 56) an approach that is the obvious beginning to such a development. Consider the following partial differential equation in two independent variables (x, y):

$$\partial^2\alpha/\partial y^2 - \partial^2\alpha/\partial x^2 = u(x)\alpha.$$

To take advantage of the invariance of this equation under translations in y, write down possible solutions in Fourier Integral form,

$$\alpha(x, y) = \int \exp(-ik(x + y) + h(x, k))dk;$$

h then satisfies a Riccati equation in x, with k as parameter. The 'geometric optics' [6] asymptotic expansion of the solution of this equation then provides the basic data both for the Inverse Scattering Technique and the computation of the conservation laws for the Korteweg-de Vries equation.

There is also considerable material on the physical applications and other equations to which a similar technique applies (e.g. sine-Gordon). A further feature is a treatment of the Baecklund transformation (a main topic in Professor Lamb's own research) from the classical point of view, especially the work of Bianchi and Goursat. A new geometric point of view (which can be developed in a modern fiber bundle-connection framework) has been introduced in the work of the physicists Frank Estabrook and Hugo Wahlquist [7], but treatment of this is precluded by Lamb's decision to avoid geometry.

A lot of information is packed into 278 pages! This is essential reading both for general education about how mathematics is being applied today and for the experts who want to read an insightful review of some basic ideas.

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