RESEARCH ANNOUNCEMENTS

SPECTRAL PROPERTIES OF SOME NONSELFADJOINT OPERATORS

BY A. G. RAMM

ABSTRACT. Let $A$ be a compact linear operator on a Hilbert space $H$, $s_n(A) = \{\lambda_n(A^*A)\}^{1/2}$, $Q$ be a compact linear operator, $I + Q$ be invertible, $B = A(I + Q)$. We prove that $s_n(B)s_n^{-1}(A) \to 1$ as $n \to \infty$. If $|Qf| < c|Af|f^{1-a}$, $a > 0$, $c > 0$, $f \in H$ and $s_n(A) = C_n^{-r}(1 + Q(n^{-q}))r$, $q > 0$, then $s_n(B) = s_n(A)^{(1 + O(n^{-q}))}$, where $\gamma = \min\{q, ra(1 + n)^{-1}\}$. This estimate is close to sharp. We also give conditions sufficient for the root system of $B$ to form a Riesz basis with brackets of $H$. Applications to elliptic boundary value problems are given.

1. Notations, definitions. Let $H$ be a separable Hilbert space, $A$ and $Q$ be compact linear operators on $H$, $B = A(I + Q)$, $\lambda_n(A)$ be the eigenvalues of $A$, $s_n(A) = \{\lambda_n(A^*A)\}^{1/2}$ be the $s$-values of $A$ (singular values of $A$), $C$ be various positive constants, $\mathbb{R}^d$ be the Euclidean $d$-dimensional space, $D \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary, $L$ be a positive definite in $L^2(D)$ elliptic operator of order $l$ and $M$ be a nonselfadjoint differential operator of order $m < l$. We define $s_n(L) = \{s_n(L^{1-1})\}^{-1}$. Let $A\phi = \lambda\phi, \phi \neq 0$. With the pair $(\lambda, \phi)$ one associates the Jordan chain defined as follows: consider

(*) $A\phi^{(1)} - \lambda\phi^{(1)} = \phi$. If this equation is not solvable then one says that there are no root vectors associated with the pair $(\lambda, \phi)$. If (*) is solvable then consider the equations

(**) $A\phi^{(j)} - \lambda\phi^{(j)} = \phi^{(j-1)}, j = 1, 2, \ldots, \phi^{(0)} \equiv \phi$. It is known [1], that if $A$ is compact then there exists an integer $N$ such that (**) will not be solvable for $j > N$. In this case vectors $\phi^{(1)}, \ldots, \phi^{(N)}$ are called the root vectors associated with the pair $(\lambda, \phi)$, $(\phi, \phi^{(1)}, \ldots, \phi^{(N)})$ is called the Jordan chain associated with the pair $(\lambda, \phi)$. Consider the eigenvectors $\phi_1, \ldots, \phi_q$ corresponding to the eigenvalue $\lambda$ and all the root vectors associated with the pairs $(\lambda, \phi_p), p = 1, \ldots, q$. The linear span of the eigen and root vectors corresponding to $\lambda$ is called the root space corresponding to $\lambda$. The collection of all eigen and root vectors of $A$ is called its root system. Let us define Riesz's basis of $H$ with brackets. Let $\{f_j\}$ be a linearly independent system of elements of $H$, $\{h_j\}$ be an orthonormal basis of $H$, and $m_1 < m_2 < \cdots < m_j \to \infty$ be a

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sequence of integers. Let $H_{m}(F_{j})$ be the linear span of vectors

$$h_{m,j}, h_{m,j+1}, \ldots, h_{m-1}, (f_{m,j+1} \ldots, f_{m-1}),$$

$T$ be a linear bounded invertible operator from $H$ onto $H$, $TF_{j} = H_{m}$, $j = 1, 2, \ldots$. Then the system $\{f_{j}\}$ is called a Riesz basis of $H$ with brackets. If $m_{j} = j$ then $\{f_{j}\}$ is called a Riesz basis of $H$. If a root system of $A$ forms a Riesz basis of $H$ with brackets then we write $A \in R_{b}(H)$. If it forms a Riesz basis then we write $A \in R(H)$. The range of $A$ is denoted by $R(A)$, $\lim_{n} m_{j} = \lim_{n} n_{j}$.

2. Introduction. Two questions will be discussed: (1) When is $s_{n}(B) \sim s_{n}(A)$ and what is the order of the remainder? (2) When does $B \in R_{b}(H)$?

There are few known results connected with question (1). The results are due to H. Weyl, Ky Fan and M. G. Krein (see [2]), and the author [3]. It seems that there were no abstract results on the perturbations preserving asymptotics of spectrum with estimates of the remainder. In Theorem 1 (§3 below) such a result is given. In [2] there are some results about completeness of the root systems of certain operators. In Theorem 2 an abstract result which gives an answer to question (2) is given. In Theorem 3 some spectral properties of nonself-adjoint elliptic operators are presented. F. Browder [1, Chapter 14, Theorem 28] proved completeness of the root system of $L + M$ in $H = L^{2}(D)$. We prove that $L + M \in R_{b}(H)$ by applying Theorem 2. In order to do this note that $(L + M)^{-1} = A(I + Q)$, where $A = L^{-1}$, $Q = -(I + ML^{-1})^{-1}ML^{-1}$. During the last decade there was a great interest among physicists and engineers in question (2) and some results due to Markus, Kacnelson, Agranovich and others were used [4] (see also Appendix 10 in [3], [5], [6]).

3. Results. We will not repeat in this section the notations and assumptions of §1 but they are assumed to be valid.

**Theorem 1.** If $N(I + Q) = \{0\}$, $\dim R(A) = \infty$, then $\lim_{n} s_{n}(B)s_{n}^{-1}(A) = 1$. If $|Qf| \leq c|Af|^{a}|f|^{1-a}, a > 0$, for all $f \in H$ and $s_{n}(A) = cn^{-r}(1 + O(n^{-q}))$, $r, q > 0$, then $s_{n}(B) = s_{n}(A)(1 + O(n^{-q}))$, where $\gamma = \min\{q, ra(1 + ra)^{-1}\}$.

**Remark 1.** The estimate of the remainder is close to sharp: for the elliptic operators in $L^{2}(D)$ the remainder is of order given in Theorem 1.

**Theorem 2.** If $A > 0$, $\lambda_{n}(A) \sim cn^{-r}$ as $n \to \infty$, $r > 0$, $|Qf| \leq c|Af|$, $0 < a$, $N(I + Q) = \{0\}$, and $ra \geq 1$, then $B \in R_{b}(H)$.

**Theorem 3.** If $l - m \geq d$ then $L + M \in R_{b}(H), H = L^{2}(D)$. Furthermore
if \( N(L + M) = \{0\} \), then \( s_n(L + M) = s_n(L)\{1 + O(n^{-\gamma})\} \), where
\[
\gamma = \min\{d^{-1}, (l - m)(l - m + d)^{-1}\}.
\]

**Remark 2.** If \( d = 1 \) then \( m < l \) implies \( l - m > 1 \), and \( L + M \in R_b(H) \).

4. **Problems.** (1) Let \( Bf = \int_{-1}^{1} \exp\{i(x - y)^2\}f dy \) be an operator on \( H = L^2([-1, 1]) \). It is not known if \( B \in R_b(H) \). (2) If \( d > 1 \) it seems to be an open problem if \( L + M \in R(H) \) under the assumption of Theorem 2. Is the bracketing necessary? Some other problems can be found in [3, 5], where some questions of interest in applications are also discussed.

5. **Comments.** Minimax representation for \( s_n(B) \) is the key point in the proof of Theorem 1. A proof of Theorem 2 can be based on a result from Appendix 11 in [3]. Theorem 3 can be derived from Theorem 2 and some known estimates for elliptic operators.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MANHATTAN, KANSAS 66506