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A formal background to mathematics, by R. E. Edwards, Universitext, Springer-Verlag, New York-Berlin-Heidelberg, Volumes 1a and 1b, 1979, xxxiv + 933 pp., $29.80; Volumes 2a and 2b, 1980, xivii + 1170 pp., $39.80.

Before I can review this unusual book adequately, it is necessary to consider how academic mathematicians (such as readers of this Bulletin) have gradually come to expect mathematics books to be written in a standard way. About a century ago, for example, books on Physics were appearing in Britain, written by confident authors who declared in enthusiastic Prefaces that their readers could now understand the mysteries of the Universe, God's handiwork, etc. By the 1920's the style had changed and a different type of author would say in his Preface 'This book contains all that is necessary for the B.Sc. Examinations of . . . '. In Mathematics, as well as Physics and other disciplines, the Enthusiast still writes, but he usually conforms to the American euphemism of the 'Reward Structure'; his readership may not be working for B.Sc. degrees, but they are in a hurry and want him to get cracking. They want the gold nuggets of the mathematical mine, without the view from the mountain, or disquisitions upon the systems fed by the gold. Among other things, therefore, they are unlikely to dwell on his Preface (especially if it is long and complex), and they expect a flow of information that is organised and assembled in a conventional way, to get to the point as quickly as possible. Of course, there is also an unspoken convention as to what this 'point' is, and the convention is reinforced by almost every new book that appears in the rapid growth of mathematics publishing during the past 40 years.

A few books appear which have a different 'point', and this is one of them. Like the early Physics books, it aims to put the gold nuggets of the mathematics into a synoptic view. It is a work of Mathematics Education, which is a more complex affair than 'just' Mathematics. It began when, in a climate of widespread change in Australian mathematics curricula, the author gave a course to Australian high-school teachers. Following Felix Klein and the usual way of most academic mathematicians when they first begin such work, he thought naïvely that he would tell them what he thought they should know. In his case he wanted to talk about high-school Calculus from the point of view of his own enthusiasm for Functional Analysis. But teachers have minds of their own and often prefer to ask about what worries *them*, so Edwards was surprised to find that they wanted help from him on subtle questions of Logic and Foundations that conventional Analysis courses usually ignore. He tried to meet their questions honestly, and got worried about the adequacy of his 'working mathematician's' answers; so after wide reading and self-study he gradually gathered together some answers and tried them out on the teachers. A few of the teachers stayed to the end of a long sequence of meetings, and the fruits of the work are contained in the book under review—about 2000 pages with 16 Chapters, an extensive and useful bibliography, lengthy appendices and notes, and some 250 problems with annotations. The work is divided into two 'Volumes', each consisting of two physical volumes. It is not quite as long as it sounds, because printed from camera-ready copy, using type-script with very generous spacing. (Just to refer rapidly to editorial matters, there are a few unimportant typing errors, and the Preface and Forward have been interchanged. Also the author became aware of Rosser's book [**5**] only at a very late stage, but that—excellent though it is—was written for a different purpose and audience.)

Because a high percentage of most of the pages look (and are) highly technical, a superficial browse through Edwards's book may cause a reader to say that the author has 'merely' given a super-rigorous course of Analysis. True, he goes as far as the residue theorem, Laurent Series and the branches of Log in complex function-theory—and incidentally includes many unhackneyed examples, and nontrivial insights into Fourier analysis and differential equations on the way. Also one Appendix contains a reprint of an article by Hanna Neumann on elementary Probability theory. But such a verdict views the book solely as a supplier of mathematics in the 'nugget' tradition mentioned earlier. The book's real interest, in my view, is the way in which it grapples at several levels with what I have discussed in [**2**], [**3**] as the Problem of the Three Languages, a basic problem of mathematics education which can be stated as follows. Suppose a person A chooses to communicate a piece of mathematics to a person B, that mathematics being in a reasonably finished form in the literature. It will reside there, written in an 'official' mathematical language L_M, which may be much more refined than the working language L_P of B (who is—possibly only temporarily—in the role of pupil to A's role of teacher). As teacher, A must find a language L_T to form a bridge that allows

B to pass at least part of the way from L_P to L_M. By 'language', we mean more than just vocabulary and grammar; L_M embodies overtones of the international mathematical tradition and community and conventions about questioning, argument, and rationality which may be at variance with those inherited by B from the social traditions of his natural environment. It is part of A's task as teacher to make reliable inferences about L_P and its limitations, which will help in designing a good L_T (which, by the way, need not be an entirely literary language: gestures and other aids may be included). If A is a hard-liner, he might ignore L_T and L_P and insist that B accepts L_M or nothing; but apart from well-known inadequacies in that approach as a teaching method, A himself is likely to object to it when he is learning mathematics as a reader of other people's works.

In the situation from which Edwards (as 'A') began, the 'piece of mathematics' was a large chunk of mathematical Analysis, with an L_M that exists in several dialects. His 'B' was a group of high-school teachers with language L_P that included elementary calculus, and half-digested memories of academic mathematics from College days. When a question is asked in L_P, about worrying aspects of calculus, most University teachers will nowadays formulate an answer in some version of the language W of the 'working mathematician' for talking about functions, limits, differentials, etc. Most of us are happy with W as an adequate basis for clarifying a good deal of Classical Mathematics; it is still fashionable to use it in programs of in-service training for teachers, as in the book [4]. But suppose our pupil B is unhappy with ostensive definitions of concepts such as 'set' and 'predicate' (after which W begins to flow fairly smoothly except for artistic problems): what then? Well, you have to develop a truly formal language Φ within a metalanguage, and then show how Φ is simplified in a human, rather than mathematical, way to become first a semiformal language, and then bastardised further into what Edwards calls 'Routine Mathematics', a part of W. Simply for mutual discussion by practitioners, and excluding the usual educational constraints, we already have at this point three levels of language. To undertake all this is a long job, and Edwards sets it out as follows (modifying somewhat the approach of Bourbaki).

We begin with a primitive alphabet, and four logical signs including an alias τ for Hilbert's ε-symbol, to be used as a set-builder. Strings are formed according to certain rules, and certain of these can be picked out as 'sets' and 'sentences'. With a great deal of patient commentary, and asides about usages and abusages within W, the author gradually builds a working mathematics in the expected way, which allows us to establish the usual rules for using Sets and Functions, to build the number-systems and then on into the limiting processes of Calculus and Analysis. The logical nit-picking is gradually relaxed in favour of the more conventional worries of Analysis, but every now and then the author takes a standard proof written in the routine style of W, and gives it a full-dress revision within the framework of Φ.

In such revisions, he also points out how far we still are from absolute puritanical purity by occasionally estimating the size of the string abbreviated by a working formula and noting various tedious gaps that still need to be filled. Here, I am reminded of a conversation I once heard, when a classical analyst was laying down the law about how good it was (for the soul, I think) to assault undergraduates with full analytical rigour: a logician turned on him and said contemptuously 'You've never proved anything *properly* in your life'. One man's rigour' is another man's 'sloppiness'—or rather, these judgements reside in their respective working languages when we see how fragments of W can appear in different situations as an L_M or as an L_T.

A person who is only beginning to learn W will surely find the details of Φ difficult to read on his own, although—like Edwards's teachers—he may well find them palatable in face-to-face discussion with an experienced mentor. (The author insists several times that this is not a book for anybody meeting the material for the very first time, but his implied picture of such a person is rather inconsistent.) However, there is one portion of the book, the chapter on Complex Function Theory, where I know from experience that even personal contact has its limitations because the material is conceptually difficult. Now, the author avoids almost entirely the use of diagrams in Analysis, for the classical reason that they can mislead. But they can lead to considerable insight into the harmony of things in the complex plane that symbols obscure; and before you can understand (as distinct from check, line by line) a symbolic proof of a theorem, you need first to understand what the theorem really means. Relative beginners in W would first need (I suggest) an L_T for complex functions, that contains many diagrams, to give them algorithmic fluency in evaluating residues etc.; then they may later realise the need for more detailed proofs in an appropriate L_M. If, from the start, we attempt to remove ambiguities by inserting 'clarifying' detail, the additional stamina needed for understanding still leaves the message obscure—a basic *uncertainty principle* in mathematics education. Incidentally, the lessons of 'Geometrical' Analysis indicate that there is no one 'language of Analysis' in the missionary sense of G. H. Hardy or Whittaker and Watson.

To summarise, then, the author is concerned to show how we may pass (as if applying a 'popularising functor') from the refined but tedious language Φ to the working language W, *in order that* high-school teachers may then pass in a strong, controlled way from W to the language of their own pupils. In the process, his commentaries call into play other levels of expository language, and the result is a complicated but fascinating exposition. (I wonder if Mathematical Logicians, or Computer Scientists, could not be led to meditate on these questions of levels of language (see also Austin and Howson **[1]**): might not a Platonic, 'perfect', proof reside only in an inverse limit of a system of—suitably formalised—languages and popularising morphisms?). My own preference, however, would have been a more visible segregation between the levels of language: for some time to convey the *mathematics* in W (and comment on that) and later to introduce Φ. Pedagogical order is not usually logical order, for mathematics learning is not tidy, and passengers want to get off the bus at different stages of the journey into mathematical territory.

In some asides the author shows how W can be used to resolve ambiguities in examination schedules and questions. This is important in countries like

Australia and Britain, where it is part of the philosophy of public examinations that the questions are closed and so must not be ambiguous in any way; the onus is on those who set them to be as clear as possible in saying what the questions mean.

Also, and with humble diffidence, the author does something which is rarely if ever done elsewhere and looks at various books which deal with particular topics. He analyses statements made by them; and this type of criticism, if pursued more widely with similar courtesy and responsibility, could be good for us all for two reasons. First, the criticised authors can benefit because we have no process institutionalised in the mathematical community for deciding whether books are good or bad (there is the writing of reviews, but these rarely deal with the detailed criticism that can come from *usage* of books, and it is a standard practice that reviewers do not criticise in the manner of (say) drama-critics). Second, teachers or intending teachers need training in how to look at books, deluged as they are by publishers wanting them to adopt texts which, with a wrong choice, can result in heavy investment in an unsuitable book; yet few mathematics or education courses give any training in the skills of reading a book, let alone assessing it. These and many more of the author's asides will be valuable and absorbing for inexperienced readers attempting the book on their own, although I suspect they will find the book too difficult *in toto*, partly because the later Analysis is hard, but especially for the reasons of language-levels given above. But, the book is very suitable for use in courses—to teachers and others—where an instructor can act as guide; and several courses could be selected from the material provided here. Indeed, its main (and very considerable) value will, I believe, lie in its use as a major resource by tertiary teachers to help them produce better versions of a W for their own students. As they themselves grapple with the standards of rigour implied in the language Φ and read the honest worries revealed by the author, they may also appreciate the difficulties of their own students upon whom they so often impose W without adequate thought about an appropriate teaching language L_T.

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