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BOOK REVIEWS

Operator methods in quantum mechanics, by Martin Schechter, Elsevier North-Holland, New York, 1981, xx + 324 pp., $32.50.

The quantum theory has dominated physics for over half a century, yet it retains some peculiar features. The basic concepts are observable and state. When a system is in a specified state, each observable is a well-defined random variable. However there is no sample space on which all of these random variables are simultaneously defined, and hence no notion of the outcome of the total experiment that underlies the observation one chooses to make. In short, there is no real world [2].

Nevertheless, there is a reasonably coherent mathematical formulation of quantum mechanics, based on the theory of operators in Hilbert space. Let $\mathcal{H}$ be a Hilbert space. The inner product of ϕ and ψ in $\mathcal{H}$ is denoted $\langle\phi, \psi\rangle$. An operator is a linear transformation A from a linear subspace $\mathfrak{D}(A)$ to $\mathcal{H}$. If $\mathfrak{D}(A)$ is dense in $\mathcal{H}$, then A has an adjoint operator A^*. It is defined on the linear subspace $\mathfrak{D}(A^*)$ consisting of all ϕ in $\mathcal{H}$ such that there exists a χ in $\mathcal{H}$ (necessarily unique) with $\langle\phi, A\psi\rangle = \langle\chi, \psi\rangle$ for all ψ in $\mathfrak{D}(A)$. The operator itself is defined by $A^*\phi = \chi$. An operator A is selfadjoint if $A = A^*$. Selfadjoint operators have a spectral theory, and one consequence of this is a functional calculus that gives a natural meaning to $f(A)$ for every Borel measurable function f and every selfadjoint operator A. Thus selfadjoint operators resemble random variables in that one can form functions of them.

There is a conventional dictionary relating the physics to the mathematics. It goes like this:

observable	– selfadjoint operator A
state	– unit vector ψ (defined up to scalar multiple)
energy	– distinguished selfadjoint operator H
Schrödinger picture state	– $\psi(t) = \exp(-it\,H)\psi$
Heisenberg picture observable	– $A(t) = \exp(it\,H)A\exp(-it\,H)$
expectation of observable at time t	– $\langle\psi(t), A\psi(t)\rangle = \langle\psi, A(t)\psi\rangle$.

The choice between the Schrödinger picture and Heisenberg picture is arbitrary. In either case the problem is to compute the expectation at time t, and this reduces to the computation of the unitary time evolution operator $\exp(-it\,H)$.

In the quantum mechanics of nonrelativistic particles, the energy operator may be written $H = H_0 + V$, where H_0 and V are the kinetic and potential energy operators. (There are various known conditions that ensure that the sum $H = H_0 + V$ of selfadjoint operators is selfadjoint.) The idea is that it is easy to compute functions of H_0 or of V. However H_0 and V do not commute, so the problem of computing functions of H is far from trivial. In fact it encompasses much of physics.

The problem divides naturally into two parts, corresponding to point spectrum and continuous spectrum. The point spectrum consists of the eigenvalues of H corresponding to eigenvectors in the Hilbert space. The closed subspace spanned by these eigenvectors is called the point spectrum subspace. The theory of the point spectrum is relatively straightforward. The continuous spectrum subspace is the orthogonal complement of the point spectrum subspace. The corresponding continuous spectrum consists of eigenvalues of an extension of H to some (appropriately chosen) larger space. Thus the eigenvectors are *not* in the original Hilbert space (where are they, then?) and the theory is rather subtle. It is understandable that quantum mechanics texts tend to be rather vague about the details.

One needs a mathematics text that can be read in parallel with a conventional physics text. It should sharpen the theory, but not be so heavy that one loses contact with the physics. There are already books by Amrein, Jauch, and Sinha **[1]** and Prugovečki **[6]**, and Schechter's book is now another candidate. The multi-volume treatises by Reed and Simon **[7]** and by Thirring **[8]** are also relevant to such an enterprise.

The basic material that should be covered is the motion of a single particle in a fixed field of potential energy. (This is actually the two-body problem since it is what remains when one removes the center of mass motion for a two-body interaction. The next step up, adding another particle, is always called the three-body problem, even after the center of mass motion has been removed.) In this elementary situation the Hilbert space takes the concrete form $\mathcal{H} = \mathcal{L}^2(\mathbf{R}^n)$, where n is the dimension of space. In nature $n = 3$, but it is illuminating to keep it as a free parameter. The Laplace operator Δ is selfadjoint and $H_0 = -\Delta$ is the kinetic energy. The interaction is given by a real measurable function v and the potential energy operator V is simply multiplication by v. The fundamental problem is the relation between v and the spectral properties of $H = H_0 + V$.

At the outset Schechter makes two fundamental choices. The first is not to try to develop all the complicated lore of physics, but to introduce quantum mechanics with a sparse set of postulates. This is a sensible choice, but it does leave open the question of how easy it will be to make connections with the physics literature. His second choice is to develop the general operator theory framework, but to illustrate it only with a detailed analysis of motion in one dimension, the case $n = 1$. More will be said about the consequences of this choice later.

The book begins with an ingenious and attractive motivation of selfadjointness. A densely defined operator A is said to be Hermitian if $\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$ for all ϕ and ψ in $\mathfrak{D}(A)$. This is a natural condition for quantum mechanics, since it implies that the expectation $\langle \psi, A\psi \rangle$ is real. However it is weaker than selfadjointness; it is still possible that A^* is a proper extension of A. Postulate 3 is that an observable is represented by a maximal Hermitian operator (one with no proper Hermitian extension). This still does not imply selfadjointness. However one also wants a functional calculus. A minimal condition is that the square of the observable is represented by A^2, and this is Postulate 4. The relevant mathematical fact is now that if A and A^2 are both maximal Hermitian, then A is selfadjoint.

The selfadjoint operator of principal interest is the total energy H. The first major topic is the point spectrum. States in the point spectrum subspace are called bound states. If the potential v is not too singular and if it approaches zero at infinity in some appropriate sense, then the corresponding multiplication operator V is relatively compact with respect to $H_0 = -\Delta$. In this case the operator $H = H_0 + V$ may have strictly negative spectrum, but it must consist of eigenvalues of H. Thus every state with strictly negative energy is a bound state. Furthermore, these eigenvalues are isolated and have finite multiplicity. (If there are infinitely many they accumulate at zero.) This discrete behavior of the energy is responsible for the word "quantum" in quantum mechanics. The book gives estimates on the location of the spectrum and on the multiplicity of the eigenvalues. The spirit is that of mathematical analysis. There is no attempt to relate the estimates to the intuitive uncertainty principle of the physicists [**4**; **7**, vol. 4]. There is a problem here with the choice $n = 1$, in that the facts are qualitatively different from those in the physical dimension $n = 3$. For instance, one nice one-dimensional result is that $\int v(x)\, dx < 0$ implies that H has a strictly negative eigenvalue. However the analog for $n \geqslant 3$ is false; even if v is negative it need not create a bound state.

The remainder of the book deals with the continuous spectrum. One of the charms of quantum mechanics is that mathematical objects acquire a second meaning. One thinks of spectral theory in terms of motion of particles and continuous spectrum suggests scattering. The kinetic energy operator H_0 generates free motion and provides a useful reference system for the actual motion governed by the total energy H. Since the spectral properties of H_0 are explicitly known, it is often not difficult to prove that for every ϕ in $\mathcal{H}$, there exists $W_\pm \phi$ in $\mathcal{H}$ with

$$\text{(1)} \qquad \exp(-it\, H) W_\pm \phi - \exp(-it\, H_0)\phi \to 0 \quad \text{as } t \to \pm\infty.$$

The wave operators $W_\pm$ are isometries with ranges that are closed subspaces of the continuous spectrum subspace for H. The hard part is to get some control on these ranges. A vector in the range of W_- is a state that came in with asymptotically free motion in the past, and a vector in the range of W_+ is a state that will go out in the same way in the future. The wave operators are said to be weakly complete if their ranges are equal. Call this common range the scattering subspace. For every ψ in the scattering subspace there are $\phi_\pm$ in $\mathcal{H}$ with $\psi = W_\pm \phi_\pm$, so that

$$\text{(2)} \qquad \exp(-it\, H)\psi - \exp(-it\, H_0)\phi_\pm \to 0 \quad \text{as } t \to \pm\infty.$$

In this case the scattering operator defined by

$$\text{(3)} \qquad S\phi_- = \phi_+$$

is unitary. What goes in must come out (and vice versa).

The book also treats a stronger notion of completeness, in which the scattering subspace is required to be the entire continuous spectrum subspace. This says that everything that is not bound is scattered, and this is certainly what one would hope for in a normal two-body problem. The problem is to find out for which interactions v it is actually true. One would also like to know how to compute the scattering operator S, since it is the object of experimental interest. Throw something in; at what angle will it come out (that is, with what probabilities)?

The book develops the relevant scattering theory in a functional analysis setting. It actually presents two treatments of the problem. The first is to Fourier transform from the time variable t to the energy variable λ. The physically relevant real values of λ are regarded as limiting values of λ from the upper or lower complex half-plane. The technique is to use theorems about the boundary values of analytic families of compact operators. The second treatment is based on a theory of perturbations by trace class operators. In both treatments the apparatus is applied to one-dimensional motion, and it is shown, for example, that if $\int |v(x)|\, dx < \infty$, then the wave operators are weakly complete. (There are also results on the stronger form of completeness.) Here the choice to work with one-dimensional motion is particularly curious. The operator techniques are natural to quantum mechanics, and they give more or less the same results in any number n of dimensions. On the other hand, when $n = 1$ the problem reduces to integrating an ordinary differential equation, and this fact may be used to give a more elementary approach. Ordinary differential equation techniques may be artificial in this context, but they will not just go away. They also work on rotationally symmetric problems in n dimensions, and they are essential to the solution of the inverse scattering problem in this case. The reader of this book may need to be reassured that the operator theory has a wider range of application than indicated by the example of one-dimensional motion.

Another point to note is the use of the energy variable. Recent approaches to the completeness problem, due to Lavine, Enss, and others [7] emphasize the time variable. There is a paradox here. The time variable is the most natural from a geometrical point of view, and the formulas are simpler to interpret. On the other hand, physicists always use the energy variable for computations. It is the natural variable for spectral analysis, and it permits localization in energy. For instance, the computational formula for the S operator (not present in the book) is

$$(4) \quad S = 1 - 2\pi i \int \delta(H_0 - \lambda)\left[V - V(H - \lambda - i0)^{-1} V\right]\delta(H_0 - \lambda)\, d\lambda.$$

If ϕ is a state with kinetic energy in a Borel set Λ (that is, an eigenvector of the spectral projection $1_\Lambda(H_0) = \int_\Lambda \delta(H_0 - \lambda)\, d\lambda$), then the only contribution to $S\phi$ comes from the integral over Λ. Experiments are performed in a fixed energy range, so it is nice to be able to compute $(H - \lambda - i0)^{-1}$ only over the desired range of λ.

A persistently vexing question is how to treat the eigenvectors corresponding to the continuous spectrum. (Recall that they do not sit in the Hilbert space.) Schechter's solution is to obtain them as a corollary of scattering theory. He finds them in $\mathfrak{L}^\infty$, which confirms the physicists's prejudices about the appropriate boundary condition, but doesn't seem to fit very nicely into the Hilbert space framework. In physics texts these non-normalizable eigenvectors occur right at the beginning and play a fundamental role throughout. A few of their properties may be derived from a simple theory based on Hilbert-Schmidt operators [3], but a detailed study seems to need scattering theory.

Schechter's book also contains a treatment of certain severe local singularities of v. It is proved that even in this situation the wave operators are weakly complete. This is reasonable, since the main factor affecting scattering should be the behavior of the potential near infinity. However there is an example due to Pearson [**5**; **1**, p. 167] that shows that it is possible for a wild enough local singularity to trap an incoming particle. Completeness of the wave operators is not a matter of mere formal manipulation; it requires serious analysis. One version of this analysis is provided in Schechter's book. In quantum physics the real world may be elusive, but some of the mathematics is now under control.

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Formal groups and applications, by Michiel Hazewinkel, Pure and Applied Mathematics Series, Academic Press, New York, 1978, xxiv + 574 pp., $52.50.

Formal groups are Lie groups treated in the style of the eighteenth century. This means, first of all, that there is no fuss about degrees of differentiability or global topology. We simply have a neighborhood of the origin in n-space with a "group law" composition $z = f(x, y)$ where the coordinates $z_i = f_i(x, y)$ are power series in the coordinates of x and y. The composition