theory. He finds them in $\mathfrak{L}^\infty$, which confirms the physicists's prejudices about the appropriate boundary condition, but doesn't seem to fit very nicely into the Hilbert space framework. In physics texts these non-normalizable eigenvectors occur right at the beginning and play a fundamental role throughout. A few of their properties may be derived from a simple theory based on Hilbert-Schmidt operators [3], but a detailed study seems to need scattering theory.

Schechter's book also contains a treatment of certain severe local singularities of v. It is proved that even in this situation the wave operators are weakly complete. This is reasonable, since the main factor affecting scattering should be the behavior of the potential near infinity. However there is an example due to Pearson [5; 1, p. 167] that shows that it is possible for a wild enough local singularity to trap an incoming particle. Completeness of the wave operators is not a matter of mere formal manipulation; it requires serious analysis. One version of this analysis is provided in Schechter's book. In quantum physics the real world may be elusive, but some of the mathematics is now under control.

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WILLIAM G. FARIS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 1, January 1982

0002-9904/82/0000-0241/$02.00

Formal groups and applications, by Michiel Hazewinkel, Pure and Applied Mathematics Series, Academic Press, New York, 1978, xxiv + 574 pp., $52.50.

Formal groups are Lie groups treated in the style of the eighteenth century. This means, first of all, that there is no fuss about degrees of differentiability or global topology. We simply have a neighborhood of the origin in n-space with a "group law" composition $z = f(x, y)$ where the coordinates $z_i = f_i(x, y)$ are power series in the coordinates of x and y. The composition should be associative wherever all the products make sense, which says that

$$f(f(u, v), w) = f(u, f(v, w)) \tag{1}$$

is identically true for the power series. We normalize by taking the origin at the identity, so

$$x = f(x, 0) = f(0, x). \tag{2}$$

Inverses need not be mentioned, because a power series inversion produces them.

At this stage we have got back to the nineteenth century, but now we take a further step and forget about convergence questions. We can define a *formal group law* by requiring properties (1) and (2), and for this it makes no difference whether the power series actually converge anywhere. As one might easily predict, it is rather recursion relations and number-theoretic properties of the coefficients that play the major role in the study of formal groups. Euler would have felt completely at home.

Nonetheless, the study of formal groups as such did not begin until a 1946 paper by Bochner [**1**]. His purpose was to show how "formal" the usual Lie group classification was. Working over an arbitrary field of characteristic zero, he constructed the Lie algebra of a formal group (you can get it from the second-degree terms) and proved that it determines the formal group up to formal change of variable. At the time this must have seemed interesting but not significant, and the paper got only four lines in *Mathematical Reviews*. Still there is one development worth tracing along this line of thought. Lie algebras of course were known to correspond to ordinary Lie groups (up to local isomorphism), and so Bochner's theorem implies that in any real formal group you can make a change of variable to get a nontrivial radius of convergence. If you wonder how to make this explicit, you will probably remember the classical "Campbell-Hausdorff formula"; for each Lie algebra it gives an associated formal group law, and it is known to converge. But now if you are thinking of arbitrary fields of characteristic zero, you may well remember that the reals are only one of the completions of the rationals—there are also all the p-adic completions. And sure enough, the Campbell-Hausdorff formula also has a nontrivial p-adic radius of convergence. Thus the whole power series theory goes over to an analogous theory of p-adic Lie groups. And in the sixties, M. Lazard [**3**] was even able to prove an analogue of Hilbert's Fifth Problem, an intrinsic characterization of those topological groups that arise as such p-adic Lie groups.

The subject really takes wings when we consider more general coefficients in $f(x, y)$. Take for instance the simplest case, where there is just one variable. If our ring of coefficients contains the rationals, then Bochner's argument shows that the Lie algebra determines the group, so up to change of variable there is only one of them. But over the integers, or in characteristic p, there is (for instance) no exponential series and thus no integral change of variable taking the formal additive group $f(x, y) = x + y$ to the formal multiplicative group $f(x, y) = x + y + xy$. (Yes, this *is* multiplication: remember the identity is at the origin, so x is the coordinate of the number $1 + x$.) Over an algebraically closed field of characteristic p, Lazard [2] showed that these two are the first two in an infinite sequence of different one-parameter group laws classified by the lowest power of x occurring in the formal product of x by itself p times.

Lazard's argument for this used the straightforward approach of taking a polynomial satisfying (1) up to degree n and seeing how it might be extended to work in the next degree. In principle, this method might be applied over any coefficient ring; one can thus prove for instance that the one-parameter formal groups over reduced rings are commutative. (This is not trivial, and it may fail when coefficients are allowed to be nilpotent.) But what the method really leads to is the *universal* one-parameter group. That is, suppose we write down a power series $f_0(x, y) = \Sigma a_{ij} x^i y^j$, leaving the a_{ij} indeterminate. As we work out the conditions that f_0 be a commutative formal group law, we find that they are a collection of polynomial identities that the a_{ij} must satisfy. We can define a ring R_0 to be $\mathbf{Z}[\{a_{ij}\}]$ where we impose just these identities on the a_{ij}. Then f_0 is a group law with coefficients in R_0, and any commutative group law f with coefficients in any ring R comes from f_0 by a homomorphism $R_0 \to R$ (send the formal a_{ij} in R_0 to whatever the actual coefficients of f are in R). Obviously then we can prove various properties and identities once and for all if we can prove them for the universal group law f_0. But what makes f_0 really interesting is Lazard's proof that R_0 is actually a polynomial ring $\mathbf{Z}[\{U_i\}]$. This fact has immediate implications; for instance, that every group over the finite field $\mathbf{Z}/p\mathbf{Z}$ is the reduction of one over the integers. And unlike some power series proofs, this one is nontrivial; that is, one cannot simply express some of the a_{ij} in terms of others that are free. The fourth indeterminate first occurs in the form $2U_4$, for instance, and only later, more complicated terms show that this element of R_0 is the double of another one.

These last remarks have been rather technical, and "the coefficient ring for the universal one-parameter commutative formal group law" does not sound like a topic apt to stir the souls of any but devoted algebraists. But in 1969 Quillen passed this way [5], and a swarm of topologists followed. It turns out that certain extraordinary cohomology theories define formal groups, and the one defined by complex cobordism is precisely the universal one. Brown-Peterson cohomology in particular arises from it by a "p-typification" process that was already familiar algebraically. Purely algebraic computations on the universal group imply that the only relations satisfied by Chern numbers of complex manifolds are those that follow from the Riemann-Roch theorem.

More could be said about this topic, but not by me. Let me retreat to a quite different application of formal groups. If we want to study algebraic extensions of a field K, the simplest ones we can begin with are those with abelian Galois group. The simplest K to work with (beyond finite fields) are the finite extensions of the p-adics $\mathbf{Q}_p$. (The theory is then called "local class field theory".) For $\mathbf{Q}_p$ itself the abelian extensions are all given by adjoining roots of unity. For larger K this is not true, and the classical theory gives only an indirect construction of the extensions. More specifically, the roots of unity of order prime to p still give all the extensions of one type ("unramified" ones), but the p-power roots of unity no longer give all the "totally ramified" extensions. In 1965 Lubin and Tate [4] discovered that we were simply looking at the wrong group. The roots of unity, after all, are just the torsion elements in the multiplicative group. There is a natural construction that gives the multiplicative group over $\mathbf{Q}_p$ but different formal groups over larger K, and they define p-adic Lie groups where the points of p-power order give precisely the extensions we need.

What next? More variables, of course—all this has been just one-parameter groups. Historically, indeed, much of the interest in formal groups arose in this generality. In characteristic zero, Lie algebras are a good way to deal with Lie groups, and in particular with algebraic groups. The algebraic groups still make sense in characteristic p, but the correspondence breaks down badly. One can still define a formal group for each algebraic group, and it is a structure intermediate between algebraic groups and Lie algebras. For semisimple groups it turned out to be easier to classify the groups directly, but for abelian varieties this approach has been very significant, particularly in number-theoretic applications.

The most striking multivariable results deal with commutative n-parameter formal groups over an algebraically closed field (of characteristic p—in characteristic zero they are all vector groups, since their Lie algebras are commutative). These, with certain natural "Frobenius" operations on them, correspond to modules over a certain noncommutative ring. This fact goes back to a string of papers by Dieudonné in the fifties, but it has been approached in a number of different ways since then, each connecting the result with different material. Dieudonné's original method worked with the "hyperalgebra", the continuous dual to the power series ring—what analysts would call the distributions on the group supported at the origin. This is a cocommutative Hopf algebra, and Dieudonné's ideas have been taken up in general Hopf algebra studies. Gabriel, approaching the subject through a different dualization, interpreted the hyperalgebras as rings of functions on commutative unipotent algebraic groups, and he classified these by their maps to "Witt vector" groups. Cartier introduced a method using "p-typical" curves, working directly with the power series; this in principle works over arbitrary base rings, and using it he was able to analyze the formal groups over p-adic integer rings with a given reduction to the residue field. Research in this area is still active.

Without explicitly saying so, I have been discussing Hazewinkel's book, because it contains everything I have mentioned (except Lie groups) and much more—I haven't found space for the nice properties of "logarithms" of integral group laws, or for generalized Witt vectors and the Artin-Hasse exponential, or for the Atkin-Swinnerton-Dyer congruences for elliptic curves, or . . . . A brief outline will show specialists where the book's emphasis lies. The text begins with 90 pages on constructing formal groups, first with one parameter and then with more; this includes the Lubin-Tate groups, but is mainly devoted to universal groups. (There is a technical innovation here, a very general formulation of the arguments for showing that solutions of certain functional equations have integral coefficients.) Then come 50 pages on curves and Witt vectors (from here on commutativity is almost always assumed). The heart of the book is two long chapters on classification theory: 160 pages on endomorphisms and classification by power series methods, and 110 pages on Dieudonné modules (done *via* curves, and including the "tapis de Cartier"). The applications then take just 50 pages, followed by 40 pages sketching the dual approach through hyperalgebras.

The one objection to the book, perhaps, is that it is sometimes a bit too reminiscent of the eighteenth century: my own tolerance for power series computations gives out before the author's does. His bent is also away from my own main interest, the connections with finite group schemes and p-divisible group schemes and abelian varieties; for this any interested reader should consult [**6**]. The three applications, though they should be clear to specialists, presume enough background that the general reader may have trouble appreciating them. Still, the book will clearly be a standard and valuable reference work. It has good bibliographical notes and a bibliography essentially complete through 1976+ . Hazewinkel has also nicely made the first section of each chapter a self-contained introduction and summary, so that anyone reading just these parts can get a fair first impression of the material. This might well be worth doing, if only to see what a sophisticated theory can arise from the simple question of which power series satisfy $f(x, f(y, z)) = f(f(x, y), z)$.

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WILLIAM C. WATERHOUSE

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 6, Number 1, January 1982
© 1982 American Mathematical Society
0002-9904/82/0000-0249/$01.75

Ramsey theory, by Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer, John Wiley and Sons, Inc., New York, 1980, ix + 174 pp., $21.95.

In recent years there has been among mathematicians an explosion of interest in discrete mathematics. In particular, an enormous number of Ramsey type results have been published during the last ten years. Thus, it is appropriate that a book should be published on Ramsey theory.

Although the scope of the book is broad, covering all of the main areas of the theory, the authors do not attempt to make the book an encyclopedia of known theorems, proofs and conjectures. Instead, they have chosen to give