

POINCARÉ AND LIE GROUPS

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When I was invited to address this colloquium, the organizers suggested that I talk on the Lie-theoretic aspects of Poincaré's work. I knew of the Poincaré-Birkhoff-Witt theorem, of course, but otherwise was unaware of any contributions that Poincaré might have made to the general theory of Lie groups, as opposed to the theory of discrete subgroups. It thus came as a surprise to me to find that he had written three long papers on the subject, in addition to several short notes. He evidently regarded it as one of the major mathematical developments of his time—the introductions to his papers contain some flowery praise for Lie—but one probably would not do Poincaré an injustice by saying that in this one area, at least, he was not one of the main innovators. Still, his papers are intriguing for the glimpse they give of the early stages of Lie theory. Perhaps this makes a conference on the work of Poincaré an appropriate occasion for some reflections on the origins of the theory of Lie groups.

Sophus Lie (1842–1899) developed his theory of *finite continuous transformation groups*, as he called them, in the years 1874–1893, in a series of papers and three monographs. To Lie, a transformation group is a family of mappings

$$(1a) \quad y = f(x, a),$$

where x , the independent variable, ranges over a region in a real or complex Euclidean space; for each fixed a , the identity (1a) describes an invertible map; the collection of parameters a also varies over a region in some \mathbf{R}^n or \mathbf{C}^n ; and f , as function of both x and a , is real or complex analytic. Most importantly, the family is closed under composition: for two values a, b of the parameter, the composition of the corresponding maps belongs again to the family, i.e.,

$$(1b) \quad f(f(x, a), b) = f(x, c),$$

with

$$(1c) \quad c = \varphi(a, b)$$

depending analytically on a and b , but not on x . It must be noted that these identities are only required to hold locally; in present-day terminology, (1a–c) define the germ of an analytic group action.

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Differentiating (1a) with respect to the coordinates a_i of the parameter, Lie constructs vector fields

$$(2) \quad X_i(F) = \sum_j \xi_{ij}(x) \frac{\partial F}{\partial x_j}$$

(Lie's notation), which he calls the *infinitesimal transformation* of the family. That is how he pictures them and, on occasion, calculates with them. The foundations of Lie's theory are embodied in his *three fundamental theorems*. The first consists of a differential equation, involving the X_i , which is equivalent to the group property—an infinitesimal version of (1b). This turns out to be more delicate than one might expect since Lie, initially at least, does not insist on the existence of an identity transformation, or of inverse transformations, within the family (1). According to the second fundamental theorem, the linear span of the X_i is closed under the Lie bracket,

$$(3) \quad [X_i, X_j] = \sum c_{ijk} X_k.$$

Conversely, any Lie algebra (Hermann Weyl's terminology!) of vector fields generates a group in Lie's sense. His arguments are those that one would use today: the commutators in the Lie algebra correspond infinitesimally to commutators in the group, which leads to the identities (3). On the other hand, the one parameter groups generated by a collection of vector fields X_1, \dots, X_n fit together as a family, (locally) closed under composition, precisely when the X_i span a Lie algebra.

The third fundamental theorem, finally, states that any set of structural constants $\{c_{ijk}\}$, subject to the obvious necessary conditions, arises from some Lie algebra of vector fields, and hence determines a transformation group. In other words, every (finite dimensional, real or complex) Lie algebra can be realized as a Lie algebra of vector fields. Lie proves the theorem by producing a Lie algebra of functions, with respect to the Poisson bracket, which he obtains as solutions of a system of differential equations. Some years earlier, Lie had published a shorter argument: the given structure constants $\{c_{ijk}\}$ determine vector fields

$$(4) \quad X_i = \sum c_{jik} x_j \frac{\partial}{\partial x_k},$$

and these form a Lie algebra, provided the c_{ijk} satisfy the appropriate conditions. What amounts to the same, Lie constructs the adjoint group of the group whose existence he wants to establish. If the group in question has a center of positive dimension, it is not locally isomorphic to its adjoint group, and this argument breaks down—a possibility which Lie overlooked at the time.

In Lie's development of the theory, the idea of a *group action* is of primary interest, and the group itself is relegated to a supporting role. However, one can easily recover the group itself in Lie's framework: the composition rule for the parameter (1c), $c = \varphi(a, b)$, in which a may be viewed as the variable and b as the parameter, or vice versa, is a transformation group in the sense of Lie, a group which acts (locally) simply transitively. Lie calls it the first or

second parameter group, depending on whether a or b is regarded as the variable. Now one would say that the group acts on itself by left and right translation. To Lie, with the algebraic notion of a group so very far in the background, it was not obvious that the two actions commute; in fact, he credits Engel with this observation.

Friedrich Engel (1861–1941) had been a student of Felix Klein in Leipzig, and in 1884 was sent by Klein to his friend Lie in Norway, where Engel wrote his *Habilitationschrift*. Two years later, when Lie succeeded Klein in Leipzig, Engel accompanied him. Although Engel eventually became quite active on his own, at first he seems to have limited himself mainly to functioning as Lie's sounding board and selfless secretary—an arrangement with present-day parallels. The foreword to the first volume of *Theorie der Transformationsgruppen* describes Engel's contributions as primarily linguistic, but nonetheless valuable because Lie, in his own words, did not "master any of the major languages completely". Later, in his introduction to the third volume, Lie's credits to Engel become more generous. One may well suspect that Engel influenced his teacher to a greater extent than the latter's acknowledgements suggest: loose definitions and careless mistakes occur frequently in Lie's papers before 1884, but not thereafter.

To put my brief account of Lie's three fundamental theorems into perspective, I ought to remark that the foundations of the theory of continuous groups represent only a small part of his work. Lie saw his theory as a powerful tool, with far-reaching applications to the integration theory of differential equations and to the most basic problems of geometry. He pursued these applications tirelessly, in numerous publications.

Around 1980, Friedrich Schur² (1856–1932) published two papers, in which he presented an alternate approach to the foundations of Lie's theory. His point of departure is the observation, first made by Lie, that there is a canonical choice of parameters a in (1a), namely the one for which the straight lines $t \rightarrow ta$ correspond to one parameter subgroups. As one would say now, Schur parametrizes a neighborhood of the identity in the group by a neighborhood of the origin in the Lie algebra, via the exponential map. In terms of such canonical coordinates, the composition rule (1c) also assumes a canonical form: φ can be expressed as a convergent power series, whose coefficients depend polynomially on the structure constants c_{ijk} , but which are otherwise universal—the Campbell-Hausdorff formula in disguise. Schur's power series makes sense and converges near the origin whenever the c_{ijk} satisfy the obvious conditions, i.e., skew symmetry and the Jacobi identity. In particular, this gives a new proof of Lie's third fundamental theorem, one that is much more direct and, incidentally, almost simultaneous with Lie's.

A similar procedure works for any (locally) transitive transformation group. After a linear coordinate change, some of the canonical coordinates become canonical coordinates for the isotopy subgroup at a given point, and the others coordinates for the space on which the group acts. Schur's

²Not related to Issai Schur, at least not directly.

arguments do not use the analytic nature of the transformation group; two continuous derivatives are enough. It follows that any transitive, C^2 transformation group can be made analytic by means of a suitable coordinate change, a fact which had previously been asserted by Lie, without proof, and without a specific bound on the number of derivatives. This, of course, is the origin of Hilbert's fifth problem.

It is instructive to compare Schur's mathematical style to that of Lie. Schur had been a student of Weierstraß in Berlin, and was strongly influenced by Weierstraß' insistence on rigor and logical completeness. One can almost sense his discomfort with Lie's intuitive reasoning; not once does he refer to the "infinitesimal transformations", although he uses the same letters as Lie for their coefficient functions. One of Schur's papers begins with a detailed proof of the differentiability of solutions of differential equations, as functions of the initial conditions. Elsewhere he carefully estimates the radius of convergence of a power series in several variables. To Lie, such arguments must have appeared overly complicated, and even pedantic. As Engel reports in Schur's obituary, Lie and Schur had very different ideas of what was easy and what was not.

Also around 1890, Wilhelm Killing (1847–1923) wrote a series of five papers in which he established, or came close to establishing, many of the basic structure theorems about complex Lie algebras: the existence of a Levi decomposition of Lie algebras that coincide with their own derived algebras, criteria for semisimplicity—Killing, in fact, coined the term "semisimple"—and most remarkably, the classification of simple Lie algebras. Killing had been led to the classification problem by geometric considerations, to a large extent independently of Lie's work,³ but in these five papers he generally follows Lie's terminology and notation. Killing's most important tool is the notion of a *root*, i.e., root of the characteristic equation

$$(5) \quad \det(\operatorname{ad} X - \omega) = 0,$$

which Killing writes in terms of coordinates and structure constants, of course. Here X is an element of the Lie algebra \mathfrak{g} , and $\operatorname{ad} X$ the infinitesimal inner automorphism corresponding to X ,

$$(6) \quad \operatorname{ad} X(Y) = [X, Y].$$

Lie had already considered the equation (5) when he proved that every $X \in \mathfrak{g}$ lies in a two-dimensional subalgebra, but it was Killing who first studied the root pattern and recognized it as the key to the structure of a Lie algebra.

Now, ninety years later, one can only marvel at Killing's work, especially his list of the exceptional simple Lie algebras, their dimensions and root systems, all discovered during the infancy of the subject. The exposition is flawed, however, by serious gaps and errors, and is often obscure. No wonder his contemporaries remained skeptical, until Élie Cartan, in his thesis, put the results on a solid footing.

³Cf. Hawkins' article on the origin of Killing's work.

Killing's severest critic was Lie, perhaps not only for mathematical reasons, but also because—hypersensitive, as always in his priority disputes—he felt slighted by Killing's references to his own work. Lie made a habit of reviewing the work of others on what he called, with proprietary undertones, "my theory of groups". One of the pleasures of going back to the early papers in Lie theory is to read these astute, but acerbic reviews. About certain sections of one of Killing's papers, Lie writes "... the correct theorems in them are due to Lie, the false ones due to Killing", and in a sweeping damnation of several papers by Killing, "... (they) contain not so many results that are correct and new. Proved, correct and new are even fewer". In spite of such harsh language, Lie does acknowledge the great value of Killing's results on the structure of Lie algebras. Among other targets of Lie's criticism, Schur and Maurer are reprimanded for not following the notation which Lie has so carefully chosen, which makes it difficult to see what is really new in their writings. Not even Felix Klein, his friend in earlier days, is spared. So much has been written by Klein's students and friends about the relationship between Lie and Klein, says Lie, that he feels compelled to set the record straight: "I am not a student of Klein, nor is Klein a student of mine, although the latter might come closer to the truth". He then goes on to berate Klein for various offenses.

Before turning to Poincaré, I should mention two short articles of J. E. Campbell (1862–1924), written in 1897, about products of exponentials of non-commuting operators. The opening paragraph of the second neatly describes their point of view: "If x and y are operators which obey the ordinary laws of algebra, we know that $e^y e^x = e^{y+x}$. I propose to investigate the corresponding theorem when the operators obey the distributive and associative laws, but not the commutative". The idea of exponentiating a vector field, or "infinitesimal transformation", to a "finite transformation" already appears in the work of Lie, who considers expressions like

$$f + X(f) + \frac{1}{2!} X^2(f) + \dots,$$

but does not use the exponential formalism for this purpose. As Campbell observes, the product of exponentials of two "small" vector fields X , Y is itself the exponential of a vector field Z :

$$(7) \quad e^X e^Y = e^Z;$$

this follows from the proof of Lie's second fundamental theorem. By laborious calculations he then derives a formula for Z , in terms of X , Y , repeated bracket operations and certain universal coefficients. Although he mentions the word "convergence", his version of the identity (7) has no analytic content. He cites Schur's paper because the same universal coefficients occur there, but remains silent about the close connection between (7) and Schur's proof of the third fundamental theorem.

Except for a short note on the groups of units in hypercomplex systems,⁴ Poincaré's essay "Sur les hypothèses fondamentales de la Géométrie" (1887)

⁴I.e., finite-dimensional associative algebras over \mathbf{R} .

is his first publication to mention the theory of continuous transformation groups. After Lobachevsky's description of hyperbolic geometry, the problem of characterizing physical space by suitable axioms had become one of the grand themes of nineteenth century mathematics. Poincaré approaches this problem, in the case of two-dimensional space, with the observation that Euclidean, hyperbolic and elliptic geometry have one important feature in common: their groups of motion act transitively, with one-dimensional isotropy groups. Using Lie's infinitesimal methods, he classifies the two-dimensional homogeneous spaces of three-dimensional groups, up to local equivalence. It is then a relatively simple matter to distinguish among the several possible cases by various geometric properties. The essay is elegantly written, but as Lie points out, with uncharacteristically gentle words, Poincaré seems unaware of earlier investigations of a similar nature, in particular Lie's own classification of three-dimensional (local) group actions on the plane.

Perhaps it is not purely coincidental that Poincaré returned to the theory of Lie groups only after Lie's death, with a *Comptes rendus* announcement (1899) that outlines a proof of the third fundamental theorem. The details follow a few months later, in the form of a paper dedicated to Sir George Gabriel Stokes, on the occasion of his eightieth birthday. In the interval, Poincaré must have learned of Schur's proof and Campbell's notes: he cites both, remarks that his own results, which overlap theirs to a considerable extent, are not as original as he had thought, and expresses the hope that his arguments contain enough new ideas to merit publication.

The paper begins with a discussion of the exponential formalism for vector fields. Every "infinitesimal transformation" X in a continuous transformation group exponentiates to a one parameter subgroup

$$t \rightarrow e^{tX},$$

and these one parameter subgroups generate the group. Campbell's identity (7) thus makes it possible to reconstruct the group law from the bracket operation on the Lie algebra. The third fundamental theorem follows, provided Campbell's formal series is known to converge. This, Poincaré points out, is also the basic mechanism of Schur's proof.

To give concrete meaning to the identity (7), Poincaré in effect introduces the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra \mathfrak{g} : it consists of all "symbolic polynomials", i.e., formal noncommutative polynomials in the generators of \mathfrak{g} , on which he imposes the identifications forced by the equalities

$$(8) \quad XY - YX - [X, Y] = 0,$$

with $X, Y \in \mathfrak{g}$. Because of (8), a homogeneous n th degree polynomial is equivalent, as symbolic polynomial, to a *symmetric* homogeneous n th degree polynomial, plus a polynomial of lower degree. This procedure, repeated inductively, makes every symbolic polynomial equivalent to a symmetric polynomial. Less obviously, the symmetric representative is unique—that is

the main substance of the Poincaré-Birkhoff-Witt theorem: in present-day terminology, the “symmetrization map”

$$S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$$

from the symmetric algebra of \mathfrak{g} to $U(\mathfrak{g})$ defines a linear isomorphism.⁵

Poincaré’s proof of the uniqueness of the symmetric representative is complicated and leaves much unsaid. As an illustration of its main idea, let us consider a symmetric polynomial P , of degree three, which can be made equivalent to zero without raising its degree. Expressed in terms of a basis $\{X_1, \dots, X_n\}$ of \mathfrak{g} , P takes the form

$$\begin{aligned} P = & \sum a_{ijk}(X_iX_j - X_jX_i - [X_i, X_j])X_k \\ (9) \quad & + \sum b_{ijk}X_i(X_jX_k - X_kX_j - [X_j, X_k]) \\ & + \sum c_{ij}(X_iX_j - X_jX_i - [X_i, X_j]). \end{aligned}$$

Since P is symmetric, so are its homogeneous components P_3, P_2, P_1 . On the other hand, the leading component

$$P_3 = \sum a_{ijk}(X_iX_jX_k - X_jX_iX_k) + \sum b_{ijk}(X_iX_jX_k - X_iX_kX_j)$$

vanishes when it is symmetrized; hence $P_3 = 0$. This can happen only if the six terms

$$\begin{aligned} (X_iX_j - X_jX_i - [X_i, X_j])X_k, & \quad -X_i(X_jX_k - X_kX_j - [X_j, X_k]), \\ (X_jX_k - X_kX_j - [X_j, X_k])X_i, & \quad -X_j(X_kX_i - X_iX_k - [X_k, X_i]), \\ (X_kX_i - X_iX_k - [X_k, X_i])X_j, & \quad -X_k(X_iX_j - X_jX_i - [X_i, X_j]) \end{aligned}$$

contribute equally to the first two sums in (9). Because of the Jacobi identity, these add up to

$$\begin{aligned} & X_i[X_j, X_k] - [X_j, X_k]X_i - [X_i, [X_j, X_k]] \\ & + X_j[X_k, X_i] - [X_k, X_i]X_j - [X_j, [X_k, X_i]] \\ & + X_k[X_i, X_j] - [X_i, X_j]X_k - [X_k, [X_i, X_j]], \end{aligned}$$

which is quadratic, equivalent to zero, and thus can be absorbed by the third sum in (9). In other words, P is equivalent to zero already as a second degree polynomial. With considerable effort, Poincaré carries out the analogous argument for symmetric polynomials of arbitrary degree: if such a polynomial is equivalent to zero, its degree can be reduced by one; the theorem follows by induction.

The universal enveloping algebra and Poincaré’s description of its structure were forgotten for almost forty years. In 1937, Garrett Birkhoff and Ernst Witt rediscovered Poincaré’s result independently, in its most general version, needless to say—for possibly infinite-dimensional Lie algebras, over fields of

⁵Over ground fields of nonzero characteristic the theorem must be stated slightly differently.

arbitrary characteristic. Apparently it was Cartan-Eilenberg, in their book on homological algebra, who first affixed Poincaré's name to the theorem.

To Poincaré, Campbell's formula is an identity in the universal enveloping algebra. For $X, Y \in \mathfrak{g}$, the product $(X^m/m!)(Y^n/n!)$ can be expressed uniquely as a sum of homogeneous, symmetric polynomials in the generators of \mathfrak{g} , of degree $k < m + n$,

$$\frac{X^m}{m!} \frac{Y^n}{n!} = \sum_{k < m+n} Z_{m,n,k}.$$

Hence, in a formal sense,

$$(10) \quad e^X e^Y = \sum_k Z_k,$$

where

$$(11) \quad Z_k = \sum_{m,n} Z_{m,n,k}$$

is a formal power series in the variables X and Y , with values in the space of symmetric, homogeneous, k th degree polynomials. The first term

$$(12) \quad Z_1 = \sum_{m,n} Z_{m,n,1}$$

plays a special role; for algebraic reasons, it must be a sum of repeated brackets in X and Y . Poincaré's version of Campbell's formula has three different aspects:

(a) the product (10) is an exponential series, which means that $Z_k = Z_1^k/k!$, for $k = 0, 1, 2, \dots$;

(b) the series (12) converges for small X and Y ;

(c) the series (12) can be written down explicitly.

For the purpose of proving Lie's third fundamental theorem, (c) is irrelevant, and Poincaré pays little attention to this problem, although the answer follows easily from his methods.

The main tool of his proof of (a) and (b) is a differentiated version of Campbell's formula; in symbolic notation,

$$(13) \quad e^X e^{\delta Y} = e^{X+\delta X}, \quad \text{with } \delta Y = \frac{1 - e^{-\text{ad } X}}{\text{ad } X} \delta X$$

(cf. (6)). Poincaré gives two separate derivations of the identity (13), one straightforward, by direct calculation with power series, the second a rather curious argument: If \mathfrak{g} is known to be the Lie algebra of a group, Campbell's formula and its infinitesimal analogue (13) follows from Lie's second fundamental theorem. The auxiliary Lie algebra $\tilde{\mathfrak{g}}$, with generators $\{X, X_1, \dots, X_n\}$ and relations

$$(14) \quad \begin{aligned} (a) \quad [X, X_i] &= \sum c_{ij} X_j, \\ (b) \quad [X_i, X_j] &= 0, \end{aligned}$$

is center-free for a generic choice of the constants c_{ij} , and hence is the Lie algebra of a linear group, by Lie's early proof of the third fundamental

theorem. In $\tilde{\mathfrak{g}}$, then, the identity (13) holds—even if the c_{ij} fail to be generic, as can be shown by a simple degeneration argument. To get the same statement in \mathfrak{g} , Poincaré specializes X_1, \dots, X_n to a basis of \mathfrak{g} and chooses the constants c_{ij} so that (14a) remains an equality. The second set of relations (14b) may be violated in \mathfrak{g} , of course, but the commutators which are not specified by (14a) only have a second order effect on Campbell's formula, and hence disappear from the differentiated formula (13)!

To interpret (13) as an analytic identity, Poincaré draws on the residue calculus. If Φ is a polynomial,

$$\begin{aligned} \Phi(\text{ad } X) &= \frac{1}{2\pi i} \int (\xi - \text{ad } X)^{-1} \Phi(\xi) d\xi \\ (15) \qquad &= -\frac{1}{2\pi i} \int \text{cf}(\text{ad } X - \xi) F(\xi)^{-1} \Phi(\xi) d\xi, \end{aligned}$$

with $F(\xi) = \det(\text{ad } X - \xi)$ and $\text{cf}(\dots) = \text{cofactor matrix of } (\dots)$; the integration extends over a circle, centered at the origin, and large enough to enclose the roots ξ_1, \dots, ξ_N of F . The same formula applies to any holomorphic function Φ , whose Taylor series

$$\Phi(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + \dots$$

has radius of convergence greater than

$$R_X = \max(|\xi_1|, \dots, |\xi_N|),$$

because Φ can then be approximated by polynomials on a suitable circle of integration. In this situation, the formal sum

$$\Phi(\text{ad } X) = a_0 1 + a_1 \text{ad } X + a_2 (\text{ad } X)^2 + \dots$$

converges and equals the integral (15). If Φ has no zeroes on the closed disc of radius R_X , the reciprocal series $\Phi^{-1}(\text{ad } X)$ also converges, to the inverse of the operator $\Phi(\text{ad } X)$,

$$(16) \qquad \Phi(\text{ad } X)^{-1} = \Phi^{-1}(\text{ad } X).$$

In particular, both (13) and the inverted form of that identity,

$$(17) \qquad \delta X = \frac{\text{ad } X}{1 - e^{-\text{ad } X}} \delta Y,$$

have a definite analytic meaning, the former for all X , the latter whenever the eigenvalues of $\text{ad } X$ lie inside the disc of radius 2π .

As the final step of the proof, Poincaré recovers Campbell's formula from its differentiated version by integration. Because of (13, 17), the relation

$$e^X e^{tY} = e^{Z(t)}$$

is equivalent to the differential equation

$$(18) \qquad Z'(t) = \frac{\text{ad } Z(t)}{1 - e^{-\text{ad } Z(t)}} Y,$$

with initial condition $Z(0) = X$. Any formal solution of (18) converges near $t = 0$, and hence on the interval $[0, 1]$ if only X and Y are small enough; $Z = Z(1)$ satisfies Campbell's formula (7) both formally and analytically.

I want to emphasize precisely what Poincaré proves. The identity (7) is first of all a formal identity, in the universal enveloping algebra of a specific Lie algebra \mathfrak{g} , which determines Z as a series in X , Y , and their repeated brackets; the Poincaré-Birkhoff-Witt theorem ensures the uniqueness of Z . Secondly, this formal series converges near $X = Y = 0$, to an analytic function

$$(19) \quad Z = \varphi(X, Y).$$

Both statements involve the Lie algebra structure, but make no reference to a group with Lie algebra \mathfrak{g} . Finally, the function (19) defines a simply transitive group in Lie's sense—the group property follows easily from the associativity of the multiplication in $U(\mathfrak{g})$.

The residue formula plays a more prominent role in Poincaré's arguments than my summary might suggest. The choice of method is characteristic: to Poincaré, the third fundamental theorem amounts to an application of the residue calculus, to Lie, it is the solution of a partial differential equation, and to Schur, under the influence of Weierstraß, an explicit power series in several variables!

As for the later history of the Campbell-Hausdorff formula, Baker published a new proof in 1905, very much in the spirit of Campbell's original proof, but with a more elaborate formalism taking the place of lengthy calculations. Baker saw the connection between the formula and Schur's proof of the third fundamental theorem, but he does not mention Poincaré's work on the subject. Hausdorff, a year later, said everything about the Campbell-Hausdorff formula that needs to be said. His precision and generality satisfy even Bourbaki—who faults Campbell, Poincaré and Baker for being vague. Like Campbell and Baker, Hausdorff focusses on the algebraic aspects of the formula, but he also establishes the convergence of the formal series by quoting Poincaré's argument. He recounts the previous history fully and accurately; in particular, he characterizes Poincaré's version as the specialization of his own universal identity to the enveloping algebra of a specific Lie algebra.

In 1901 and 1908 Poincaré published his last two papers on Lie groups: long, rambling discussions of such topics as the exponential and logarithm maps, the adjoint group of Lie algebra, and the relationship between a group and its adjoint group. It was Lie who first introduced the adjoint group and gave it its name. The statement that the adjoint group is the homomorphic image of any group with the same Lie algebra amounted to little more than a tautology for Lie, since he defined the notion of homomorphism purely in terms of the structure constants. Poincaré, on the other hand, displays the adjoint homomorphism as a map, with algebraic properties, by writing down formulas like

$$(20) \quad e^{-X} Y e^X = -\frac{1}{2\pi i} \int e^{-\xi F(\xi)}^{-1} \text{cf}(\text{ad } X - \xi) Y d\xi.$$

It must be borne in mind that Poincaré adopts Lie's definition of continuous group; the natural domain of the adjoint homomorphism is the *parameter group*: the set of symbols $\{e^X | X \in \mathfrak{G}\}$, endowed with the (local) composition rule (19).

The linear transformations $Y \mapsto e^{-X}Ye^X$, which generate the adjoint group, visibly preserve the Lie algebra structure of \mathfrak{g} . From this observation Poincaré deduces certain results of Killing about the root space decomposition. For example, if $Y_1, Y_2 \in \mathfrak{g}$ correspond to roots ω_1, ω_2 of X , the bracket $[Y_1, Y_2]$ belongs to the root $\omega_1 + \omega_2$, or vanishes if $\omega_1 + \omega_2$ fails to be a root—not a deep fact, but less evident before the algebraic notion of the adjoint homomorphism became common currency.

As a complex analyst, Poincaré is tempted to approach various global questions from the point of view of analytic continuation. The formula (20) exhibits the adjoint homomorphism as a holomorphic map; if \mathfrak{g} is center-free, this map has an inverse near $X = 0$ (the logarithm map of the adjoint group, in effect), which can be analytically continued. Similarly, the composition rule (19) is defined initially near $X = Y = 0$, but extends to a multiple-valued map: the identity $e^Z = e^Xe^Y$ can be solved locally for Z in terms of X and Y , provided \exp has maximal rank at Z ; φ is well behaved near such points, and ramifies at others. The roots of Killing's equation (5), finally, are multiple-valued functions on the Lie algebra. Poincaré studies the analytic continuations of these three types of functions in great detail, which quickly leads him to consider the center of the group—a vexing problem at a time when the notion of a global Lie group had not been defined.

The discussion of the center and of the multiple-valued composition rule make Poincaré's papers on Lie groups frustrating for a present-day reader. Bourbaki, in his historical notes, dismisses them as hastily written; he criticizes Poincaré for asserting in some places that the exponential map is surjective, and elsewhere giving counterexamples. However, the apparent contradiction can be resolved. When Lie talks of a "group with structure constants c_{ijk} ", he means a transformation group; he refers to the underlying (local) abstract group as the parameter group. Poincaré accepts this convention in principle, but in practice considers only the parameter group, the adjoint group, and sometimes also other linear realizations. Unlike Lie, he regards the parameter group as a global object, which is canonically attached to the Lie algebra; indeed, the parameter group really *is* the Lie algebra, equipped with the multiple-valued composition rule (19). The parameter group plays the role that one assigns now to the universal covering group: it maps homomorphically to any (transformation) group with the same Lie algebra—just another way of saying that it is the underlying abstract group of such a transformation group. In this setting, homomorphisms may have singularities and need not be globally defined. Thus it can happen that the exponential map of a linear group fails to be surjective, even though its underlying abstract group, or parameter group, has a surjective exponential map as a matter of definition.

One of Poincaré's examples is especially revealing. He considers two infinitesimal rotations X, Y in the group $SO(3)$, about axes l_X, l_Y in general position, with X normalized so that e^X represents a full rotation, through an angle 2π . In $SO(3)$, e^X equals the identity, but not in the parameter group. The product $e^Xe^Ye^{-X}$ is a rotation about the axis $e^Xl_Y = l_Y$, through the same angle as e^Y , hence

$$e^Xe^Ye^{-X} = e^Y.$$

On the other hand, the two rotations $e^Y e^X e^{-Y}$ and e^X have unequal axes in general, which means that

$$e^Y e^X e^{-Y} \neq e^X,$$

as elements of the parameter group. A paradox, but not a contradiction: the group laws are only required to hold locally, after all.

A satisfactory explanation became possible when the concepts of global Lie group and universal covering group were introduced. Both of these had to await the definition of a manifold, which appeared implicitly, at least, in Hermann Weyl's *Die Idee der Riemannschen Fläche* in 1913, a year after Poincaré's death.

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