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A II_1 FACTOR WITH TWO NONCONJUGATE CARTAN SUBALGEBRAS

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A maximal abelian subalgebra A of the von Neumann algebra M is called a *Cartan subalgebra* if the normalizer $N(A) = \{\text{unitaries } u \in M \text{ with } uAu^* = A\}$ generates M as a von Neumann algebra (see [4]). It is a corollary of the paper by Connes Feldman and Weiss [2] that any two Cartan subalgebras of the hyperfinite type II_1 factor R are conjugate by an automorphism of R. In this paper we construct the first example of a (separable) II_1 factor with two nonconjugate Cartan subalgebras, showing that the correspondence between ergodic theory (up to orbit equivalence) and von Neumann algebras is many to one.

The property we shall use to distinguish two different Cartan subalgebras will be the existence of asymptotically invariant sequences (see [7]): if G is a group of automorphisms of the abelian von Neumann algebra A with finite invariant trace τ, we say that G has asymptotically invariant (a.i.) sequences if there is a sequence p_n of projections in A, $\tau(p_n) = \frac{1}{2}$ and $\lim_{n\to\infty}\|p_n - g(p_n)\|_2 = 0$ for all $g \in G$. The full group [G] is the group of those automorphisms of A which coincide on some partition $\{p_g\}$ of the identity in A with elements of G. Two actions of groups G and G' on A are called orbit equivalent if there is an automorphism θ, preserving τ, for which $\theta[G]\theta^{-1} = [G']$. The existence of a.i. sequences is an invariant of orbit equivalence.

If A is a Cartan subalgebra of the II_1 factor M, the normalizer $N(A)$ acts on A by conjugation. Thus the existence of a.i. sequences for this action is a conjugacy invariant for A. So it suffices to exhibit a II_1 factor with two Cartan subalgebras, one with a.i. sequences and one without. To do this we use the well-known fact (see [3]) that if the Cartan subalgebra arises from the group-measure space construction with a countable group G acting on a space X with $A = L^\infty(X, \mu)$, then $N(A)$ induces $[G]$ on A.

We begin by noting that if G is countable with Kazhdan's property T then no ergodic finite measure-preserving action of G on A has a.i. sequences. For if

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it did, $2p_n - 1$ would be almost fixed unit vectors in the orthogonal complement of 1 in $L^2(A, \tau)$. (See [1].)

Now let H be a finite nonabelian group and let $X = \Pi_{i \in \mathbf{N}} H$ be the product measure space with normalized counting measure on H and consider the coordinatewise action of $\bigoplus_{i \in \mathbf{N}} H$ on X. Now choose for each $g \in G$ (G a property T group e.g. $SL(3, Z)$); a copy X_g of X and form $Y = \Pi_{g \in G} X_g$ with product measure μ. Both $\bigoplus_{i \in \mathbf{N}} H$ and G act on Y (G by the Bernoulli shift) and these actions commute. Moreover the action of G is ergodic and free and the action of $\bigoplus_{i \in \mathbf{N}} H$ is free. Thus the action of $K = (\bigoplus_{i \in \mathbf{N}} H) \oplus G$ on $L^\infty(Y, \mu)$ is free and ergodic and there are no a.i. sequences for K. So $L^\infty(Y, \mu)$ sits as a Cartan subalgebra in the crossed product M without a.i. sequences for its normalizer.

On the other hand, there are two central sequences g_n and h_n in $\bigoplus_{i \in \mathbf{N}} H$ with $g_n h_n g_n^{-1} h_n^{-1} \neq \mathrm{id}$ and $\lim_{n \to \infty} g_n = \mathrm{id}$ and $\lim_{n \to \infty} h_n = \mathrm{id}$ on $L^\infty(Y, \mu)$. (It suffices to take g_n and h_n "further out" in $\bigoplus_{i \in \mathbf{N}} H$.) This means that, in the crossed product M, g_i and h_i give rise to noncommuting central sequences in the sense of McDuff [6]. By her result, $M = M \otimes R$.

The game is up for one can exhibit a Cartan subalgebra B of R with an a.i. sequence $\{p_n\}$ for its normalizer (e.g. by Rohklin's theorem). But then $\{1 \otimes p_n\}$ will be an a.i. sequence in the Cartan subalgebra $L^\infty(Y, \mu) \otimes B$ of $M \otimes R$.

Note. Property T was not essential in this construction – any nonamenable group will do – see [5]. The authors would like to thank K. Schmidt for initiatting this study.

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