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THE MULTIPLICITY OF EIGENVALUES

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There are many examples of first order $n \times n$ systems of partial differential equations in 2 space variables with real coefficients which are strictly hyperbolic; that is, they have simple characteristics. In this note we show that in 3 space variables there are no strictly hyperbolic systems if $n \equiv 2(4)$. Multiple characteristics of course influence the propagation of singularities. For a different context see Appendix 10 of [2].

$\mathcal{M}$ denotes the set of all real $n \times n$ matrices with real eigenvalues. We call such a matrix *nondegenerate* if it has *n distinct* real eigenvalues.

THEOREM. *Let A, B, C be three matrices such that all linear combinations*

$$\alpha A + \beta B + \gamma C, \tag{1}$$

α, β, γ *real, belong to* $\mathcal{M}$. *If* $n \equiv 2$ (mod 4), *then there exists* α, β, γ *real,* $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ *such that* (1) *is degenerate.*

REMARK 1. The theorem applies in particular to A, B, C real symmetric.

REMARK 2. The theorem shows that first order hyperbolic systems in three space variables of the indicated order always have some multiple characteristics.

PROOF. Denote by $\mathcal{N}$ the set of nondegenerate matrices in $\mathcal{M}$. The normalized eigenvectors u of N is $\mathcal{N}$,

$$Nu_j = \lambda_j u_j, \quad |u_j| = 1, \quad j = 1, \ldots, n,$$

are determined up to a factor ± 1.

Let $N(\theta)$, $0 \leqslant \theta \leqslant 2\pi$, be a closed curve in $\mathcal{N}$. If we fix $u_j(0)$, then $u_j(\theta)$ can be determined uniquely by requiring continuous dependence on θ. Since $N(2\pi) = N(\theta)$,

$$u_j(2\pi) = \tau_j u_j(0), \qquad \tau_j = \pm 1. \tag{2}$$

Clearly

(i) Each τ_j is a homotopy invariant of the closed curve.

(ii) Each $\tau_j = 1$ when $N(\theta)$ is constant.

Suppose now that the theorem is false; then

$$N(\theta) = \cos\theta A + \sin\theta B \tag{3}$$

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is a closed curve in $\mathcal{N}$. Note that $N(\pi) = -N(0)$; this shows that

$$\lambda_j(\pi) = -\lambda_{n-j+1}(0) \quad \text{and} \tag{4}$$
$$u_j(\pi) = \rho_j u_{n-j+1}(0), \qquad \rho_j = \pm 1.$$

Since the ordered basis $\{u_1(\theta), \ldots, u_n(\theta)\}$ is deformed continuously, it retains its orientation. Thus the ordered bases

$$\{u_1(0), \ldots, u_n(0)\} \quad \text{and} \quad \{\rho_1 u_n(0), \ldots, \rho_n u_1(0)\}$$

have the same orientation. For $n \equiv 2 \pmod 4$, reversing the order reverses the orientation of an ordered base; this proves that

$$\prod_1^n \rho_j = -1.$$

This implies that there is a value of k for which

$$\rho_k \rho_{n-k+1} = -1. \tag{5}$$

Next we observe that $N(\theta + \pi) = -N(\theta)$; it follows from this that $\lambda_j(\theta + \pi) = -\lambda_{n-j+1}(\theta)$ and by (4) that

$$u_j(2\pi) = \rho_{n-j+1} u_{n-j+1}(\pi).$$

Combining this with (4) we get that $\tau_j = \rho_j \rho_{n-j+1}$. By (5), $\tau_k = -1$; this shows that the curve (3) is not homotopic to a point.

Suppose that all matrices of form (1), $\alpha^2 + \beta^2 + \gamma^2 = 1$, belonged to $\mathcal{N}$. Then since the sphere is simply connected the curve (4) could be contracted to a point, contradicting $\tau_k = -1$.

See [1] for related matters.

ADDED IN PROOF. S. Friedland, J. Robbin and J. Sylvester have proved the theorem for all $n \equiv \pm 2, \pm 3, \pm 4 \pmod 8$, and have shown it false for $n = 0$, $\pm 1 \pmod 8$. They have further results involving linear combinations of more than 3 matrices.

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