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LORENTZIAN FORMS FOR THE LEECH LATTICE

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ABSTRACT. Using recent results about holes in the Leech lattice we establish some Lorentzian constructions for that lattice.

Some years ago the first author and R. T. Curtis showed by a detailed (and unpublished) calculation that the set of points of the Lorentzian integer lattice $\mathbf{Z}^{24,1}$ which are perpendicular to the vector

$$t = (3, 5, 7, \ldots, 45, 47, 51 \mid 145)$$

is a copy of the Leech lattice. Our recent work [1, 3] on holes in the Leech lattice enables us to give a short proof of this fact. We work instead in the hyperplane of vectors $v \in \mathbf{Z}^{24,1}$ with $v \cdot t = -2$, and observe that this contains all the points mentioned in Figure 1. Two points in the figure are joined by an edge if they are distant $\sqrt{6}$, all other pairs being distant $\sqrt{4}$ apart. Since Figure 1 is a copy of the D_{24} hole diagram (see [3]) this proves the result.

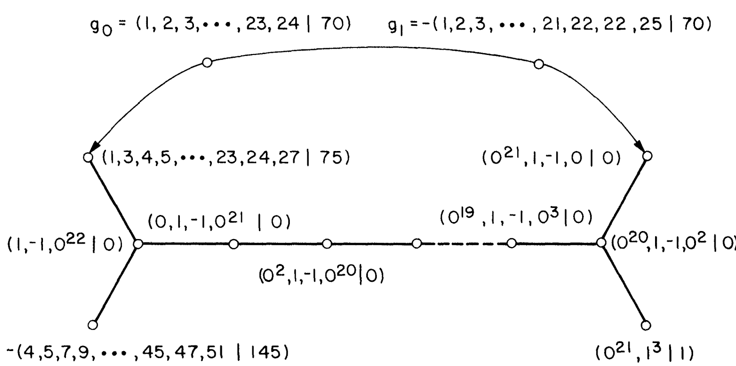

FIGURE 1. Hole diagram of type D_{24} for Leech lattice.

Seidel ([7]; see also Coxeter [4, p. 419] and Neumaier [5]) has recently remarked that elegant coordinates for the lattice E_8 may be obtained by considering the points of $\mathbf{Z}^{9,1}$ orthogonal to the isotropic vector

$$w = (1, 1, 1, 1, 1, 1, 1, 1, 1 \mid 3)$$

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modulo multiples of w. It is easy to see that we obtain the Leech lattice in a similar way from the isotropic vector

$$w = (1, 3, 5, \ldots, 45, 47, 51 \mid 145)$$

in $\mathbf{Z}^{25,1}$.

However still more elegant coordinates may be obtained as follows. We consider the even unimodular lattice L in $\mathbf{R}^{25,1}$ consisting of the points $(x_0, \ldots, x_{24} \mid x_{70})$ where the x_i are all in $\mathbf{Z}$ or all in $\mathbf{Z} + \frac{1}{2}$, and $x_0 + x_1 + \cdots + x_{24} - x_{70}$ is in $2\,\mathbf{Z}$. (Up to isomorphism there is a unique even unimodular lattice in $\mathbf{R}^{25,1}$.) We assert that the Leech lattice can be regarded as the set of all vectors of L orthogonal to

$$w = (0, 1, 2, \ldots, 23, 24 \mid 70)$$

taken modulo multiples of w. Again it is convenient to work in a parallel hyperplane, the hyperplane $H = \{v \in L : v \cdot w = -1\}$. Figure 2 shows a set of vectors in this hyperplane which contains the A_{24} hole diagram, and thus proves that $H \cap L$ is a copy of the Leech lattice.

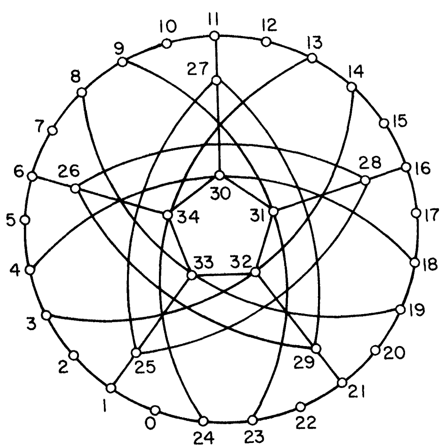

FIGURE 2. A portion of the Leech lattice contained in $H \cap L$. The coordinates of the points are as follows: $i (0 \leqslant i \leqslant 23) : (0^i, +1, -1, 0^{23-i} \mid 0)$;

$$24: \left(-\frac{1}{2}, \frac{1}{2}^{23}, \frac{3}{2} \middle| \frac{5}{2}\right);\ 25: (-1^2, 0^{23} \mid 0);\ 26: (0^7, 1^{18} \mid 4);$$

$$27: \left(\frac{1}{2}^{12}, \frac{3}{2}^{13} \middle| \frac{9}{2}\right);\ 28: \left(\frac{1}{2}^{17}, \frac{3}{2}^{8} \middle| \frac{9}{2}\right);\ 29: (0^{22}, 1^3 \mid 1);$$

$$30: (0^5, 1^{14}, 2^6 \mid 6);\ 31: (0^{10}, 1^{14}, 2 \mid 4);\ 32: (0^4, 1^{11}, 2^{10} \mid 7);$$

$$33: \left(\frac{1}{2}^{9}, \frac{3}{2}^{11}, \frac{5}{2}^{5} \middle| \frac{15}{2}\right);\ 34: (0^{14}, 1^{11} \mid 3).$$

The representatives chosen for the points v in Figure 2 have $v \cdot v = 2$, which seems to provide the simplest coordinates. However each vector in H has a unique *isotropic* representative modulo multiples of w, and using the results of [3] we have shown that

(i) if v is any isotropic vector of $H \cap L$ then $v^{\perp}/v$ is another copy of the Leech lattice, and

(ii) if v is the isotropic representative of the center of a deep hole in $H \cap L$ then $v^{\perp}/v$ is a copy of the Niemeier lattice ([2, 3, 6]) corresponding to that hole.

For instance the sum of the vectors on the 25-gon visible in Figure 2 is $(\frac{1}{2}^{25} \mid 5/2)$, which is isotropic and proportional to the center of the corresponding hole of type A_{24} in the Leech lattice $H \cap L$. Thus, for $v = (1^{25} \mid 5)$, $v^{\perp}/v$ is the Niemeier lattice of type A_{24}. This result would not be affected if, when forming $v^{\perp}$, we replaced L by the odd unimodular lattice $\mathbf{Z}^{25,1}$.

In a similar way we have shown that inside $\mathbf{Z}^{25,1}$ the vectors

$$v_1 = (1^8, 3^9, 5^8 \mid 17), \qquad v_2 = (1^{13}, 3^{12} \mid 11),$$

$$v_3 = (1^{18}, 3^7 \mid 9), \qquad v_4 = (1^{15}, 3^9, 5 \mid 11),$$

have $v_i^{\perp}/v_i$ equal to the Niemeier lattices of types A_8^3, A_{12}^2, $A_{17}E_7$, $A_{15}D_9$ respectively, and many other examples can be produced at will.

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