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FINITE LINEAR GROUPS WHOSE RING OF INVARIANTS IS A COMPLETE INTERSECTION

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ABSTRACT. The celebrated Shephard-Todd-Chevalley theorem says that for a finite linear group G operating on the n-dimensional complex vector space the ring R of invariant polynomials is a polynomial ring if and only if G is generated by pseudoreflections ($g \in G$ is a pseudoreflection if $\operatorname{rank}(g - I) = 1$). In this note we give a simple topological proof of the following statement:

If R has m generators such that their ideal of relations is generated by $m - n + s$ elements, then G is generated by those $g \in G$ such that $\operatorname{rank}(g - I) \leqslant s + 2$.

In the case $s = 0$ this gives a necessary condition for R to be a complete intersection. Our argument also gives a new simple proof of the "only if" part of the Shephard-Todd-Chevalley theorem in the case of an arbitrary ground field.

Let k be a field and let G be a finite subgroup of $GL(n, k)$. The group G acts naturally on the polynomial ring $S = k[x_1, \ldots, x_n]$ and we put $R = S^G$ to be the invariant subring of G. We say that R is a *polynomial ring* if R is generated by n (algebraically independent) elements, and that R is a *complete intersection* if R is isomorphic to $k[y_1, \ldots, y_{n+r}]/J$, where J is an ideal generated by r (= emb dim R − dim R) elements. In this paper we prove the following

THEOREM A. *If R is a complete intersection, then G is generated by the set* $\{g \in G \mid \operatorname{rank}(g - I) \leqslant 2\}$ (*where I is the identity matrix*).

The proof is based on two simple topological lemmas. We can assume that the ground field k is algebraically closed.

Let f: $\operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)$ be the quotient morphism. Let X' and Y' be the henselisations of $\operatorname{Spec}(S)$ at 0 and of $\operatorname{Spec}(R)$ at $f(0)$ respectively and f': $X' \longrightarrow Y'$ the associated morphism. Then the action of G on $\operatorname{Spec}(S)$ lifts to X' and f' is the quotient morphism. We use henselisations in order to deal with simply connected (i.e. without nontrivial étale coverings) schemes X' and Y'. If char $k = 0$, then $\operatorname{Spec}(S)$ and $\operatorname{Spec}(R)$ are simply connected and the henselisation is not necessary.

LEMMA 1. *Let Y' be a simply connected scheme, Z a closed subscheme and $Y = Y' - Z$. If Y' is a complete intersection and* codim $Z \geqslant 3$, *then Y is simply connected.*

PROOF. The proof follows from [2, X, 3.3 and 3.4].

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REMARK 1. The conclusion of Lemma 1 holds if instead of Y to be a complete intersection and codim $Z \geqslant 3$, we require that Y is regular and codim $Z \geqslant 2$.

LEMMA 2 (VINBERG). *Let X be an integral scheme and G a finite subgroup of* $\mathrm{Aut}_k(X)$. *Let $Y = X/G$ and $f: X \longrightarrow Y$ be the quotient morphism. For a closed point x of X let G_x denote the stabilizer of x. If Y is simply connected, then G is generated by all G_x's.*

PROOF. Let H be the subgroup of G generated by all G_x's; it is a normal subgroup. Then for the action of G/H on X/H any $g \neq e$ has no closed fixed points and by [1, I, 10.11], the morphism $X/H \longrightarrow Y$ is an étale covering. By our assumption, we have $G = H$.

PROOF OF THEOREM A. For $g \in G$ let L_g denote the subscheme of fixed points of g on X'. Let L be the union of all L_g's with codim $L_g \geqslant 3$, and put $X = X' - L$, $Z = f'(L)$ and $Y = Y' - Z$. Note that Y' is a complete intersection since Spec(R) is, and Z is a closed subscheme in Y' of codimension $\geqslant 3$. Furthermore, X is an integral scheme with the induced G-action, $Y = X/G$, and Y is simply connected by Lemma 1. Hence, by Lemma 2, G is generated by all G_x's, $x \in X$. But by the definition of X, $g \in G_x$ for some $x \in X$ if and only if codim $L_g \leqslant 2$ or, equivalently, $\mathrm{rank}(g - I) \leqslant 2$.

REMARK 2. R is a complete intersection for any $G \subset GL(2, \mathbf{C})$ (F. Klein). It is not difficult to construct an example of a finite group $G \subset SL(3, \mathbf{C})$ generated by two matrices A_1 and A_2, such that $\mathrm{rank}(A_i - I) = 2$, $i = 1, 2$, but R is not a complete intersection [7].

REMARK 3. Our argument together with Remark 1 gives a short topological proof of the "only if" part of the Shephard-Todd-Chevalley theorem [3, 5] over any ground field k: If R is a polynomial ring, then G is generated by pseudoreflections. It is not difficult to show that, furthermore, G_x is generated by pseudoreflections for any x. The first author takes this opportunity to suggest the following risky conjecture: Conversely, if G_x is generated by pseudoreflections for any x, then R is a polynomial ring.

If the ground field is the field $\mathbf{C}$ of complex numbers, the topology of Spec(R) is better known and we can prove the following more general theorem.

THEOREM B. *Let G be a finite subgroup of $GL(n, \mathbf{C})$ and $S = \mathbf{C}[x_1, \ldots, x_n]$. If $R = S^G$ has m generators such that their ideal of relations is generated by $m - n + s$ elements, then G is generated by those $g \in G$ such that* $\mathrm{rank}(g - I) \leqslant s + 2$.

PROOF. We set $X' = \mathrm{Spec}(S)$, $Y' = \mathrm{Spec}(R)$. By the same argument as above, we have only to prove that $Y = Y' - Z$ is simply connected if codim $Z \leqslant s + 3$, under our assumption. The corresponding generalisation of Lemma 1 in the complex case has been recently proved by Goresky and Macpherson [4].

REMARK 4. We do not know whether Theorem B is true for an arbitrary ground field.

Note, finally, that we can strengthen Theorem A (and in a similar way, Theorem B) as follows (cf. Remark 3).

THEOREM C. *If R is a complete intersection, then each G_x is generated by* $\{g \in G_x \mid \mathrm{rank}(g - I) \leqslant 2\}$.

PROOF. Let $X = \mathrm{Spec}(S)$, $Y = X/G_x$ and denote by $\pi\colon X \longrightarrow Y$ the quotient morphism. Then the morphism $Y \longrightarrow X/G$ is étale at $\pi(x)$ by [1, I, 10.11], Hence the local ring at $\pi(x) \in Y$ is a complete intersection, and we can apply Theorem A.

REMARK 5. The converse of Theorem C is false (cf. Remark 2).

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