> against this that Emmy Noether protested. What she protested against was the pessimism that shows through the last words of the quotation from Weyl's speech of 1931; the substance of human knowledge, including mathematical knowledge, is inexhaustible, at least for the foreseeable future—this is what Emmy Noether firmly believed. The "substance *of the last decades*" may be exhausted, but not mathematical substance in general, which is connected by thousands of intricate threads with the reality of the external world and human existence. Emmy Noether deeply felt this connection between all great mathematics, even the most abstract and the real world; even if she did not think this through philosophically, she intuited it with all of her being as a great scientist and as a lively person who was not at all imprisoned in abstract schemes. For Emmy Noether mathematics was always knowledge of reality, and not a game of symbols; she protested fervently whenever the representatives of those areas of mathematics which are directly connected with applications wanted to appropriate for themselves the claim to tangible knowledge. In mathematics, as in knowledge of the world, both aspects are equally valuable: the accumulation of facts and concrete constructions and the establishment of general principles which overcome the isolation of each fact and bring the factual knowledge to a new stage of axiomatic understanding.'

One must be grateful to Auguste Dick, to her competent translator, to her many helpers (e.g. Olga Taussky) and to the publisher for submitting to the English reader a lively and well-researched report on the life and work of a mathematician whose scientific influence is with us every day but whose life had become a legend already in my student days.

HANS ZASSENHAUS

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Product integration, by John D. Dollard and Charles N. Friedman, Encyclopedia of Mathematics and its Applications, Vol. 10, Addison-Wesley, Reading, Mass., 1979, xxii + 253 pp., $24.50.

The idea of product integration was first introduced by Volterra in his study of the evolution differential equation

$$dy/dt = A(t)y. \tag{1}$$

Let us consider the case where A maps an interval $[a, b]$ into the set of linear operators on a normed vector space $(X, \|\cdot\|)$ and y has the initial value

$y(a) = x \in \operatorname{Dom} A(a)$. For the purpose of developing a product integral solution, equation (1) can be approximated, following Euler's method, by a system of difference equations

$$(2) \qquad \frac{y(\tau_{i+1}) - y(\tau_i)}{\tau_{i+1} - \tau_i} = A(\tau_i)y(\tau_i) \qquad (i = 0, \ldots, n-1)$$

where τ, given by $a = \tau_0 < \tau_1 < \cdots < \tau_n = b$, forms a partition of $[a, b]$. Solving (2) recursively, we obtain

$$(3) \qquad y(\tau_k) = \prod_{i=0}^{k} [I + A(\tau_i)(\tau_{i+1} - \tau_i)]x \qquad (k = 0, \ldots, n).$$

Now under suitable hypotheses on A (e.g. if A maps continuously into the bounded linear operators on a Banach space) the finite product in (3) will converge to the *product integral*

$$(4) \qquad \prod_a^t [I + A(\tau)\, d\tau]x \qquad (a < t < b)$$

as the partition τ becomes finer and finer. This product integral will then be the unique solution $y(t)$ of (1).

The first chapter of *Product integration* constructs the product integral solution (4) for the special case where $A(\cdot)$ is continuous and matrix-valued. In this case equation (1) is just a system of n linear differential equations in n unknowns: $y_i'(t) = \sum_{j=1}^n a_{ij}(t)y_j(t)$ $(i = 1, \ldots, n)$. The product integral solution to such a system is then carefully analyzed along with questions of continuous dependence of parameters and asymptotic behavior. Thus Chapter 1 would serve well as an insightful supplement for a first course in ordinary differential equations. In the words of the authors, "...we point out the beautiful simplicities brought to the theory of linear ordinary differential equations by viewing them from the product integral viewpoint." The chapter concludes with a treatment of equation (1) when the components of $A(t)$ are merely Lebesgue integrable.

Later chapters are devoted to further relaxing the assumptions on $A(t)$. In Chapter 3 *Strong product integration*, this is achieved by having $A(t)$ act in an arbitrary Banach space X, and by requiring that $A(\cdot)x$ be strongly measurable for each $x \in X$ and that $\int_a^b \|A(\tau)\|\, d\tau$ be finite. In Chapter 5 *Product integration of measures*, the integrability of A is further weakened by requiring only that $\int_{[a,b]} \|A(\tau)\|\, \mu(d\tau)$ be finite for some positive Borel measure μ. In each of these chapters a product integral equal to (4) is formed as

$$\lim_{|\tau| \to 0} \prod_{i=0}^{k} e^{A(\tau_i)(\tau_{i+1} - \tau_i)},$$

and so is written as $\prod_a^t e^{A(\tau)d\tau}$, and is shown to provide the unique solution to equation (1).

A recurrent theme of this book is the analogy between product integration and ordinary integration. "This analogy is very far-reaching, and when trying to intuit what properties should be possessed by product integrals it is often useful to remember properties of ordinary integrals, and attempt to convert statements about additivity into corresponding statements about multiplicativity" (p. 11). Thus in Chapter 2 *Contour product integration*, the development is guided by the theory of ordinary contour integrals. Likewise, Chapters 1, 3 and 5, besides solving equation (1), concentrate on deriving for product integrals: the *Fundamental theorem of calculus*, the additivity rule, the change of variables formula, a theory of improper integration, and a variation of parameters formula. There is much overlap of topics and methods in these chapters, and by the fifth one I was experiencing a bad case of *dèjà vu*.

Chapter 4 *Applications*, includes asymptotics for the Schrödinger equation and a product-integral proof of Weyl's limit-circle classification. The chapter ends with a problem which arises in connection with continual position measurements in quantum mechanics and involves solving equation (1) for a particular linear $A(t)$ which not only is unbounded for each t but whose domain depends on t. This is indeed impressive material; it relies on a general theorem covered earlier which solves (1) in case $A(t)$ is linear, skew-adjoint, and unbounded with domain independent of t. Two other topics covered in this chapter, namely the Lie product formula and the Hille-Yosida theorem, also deserve comment. (i) The Lie product formula for matrices A and B states

$$e^{A+B} = \lim_{n\to\infty} (e^{A/n}e^{B/n})^n. \tag{5}$$

By introducing product integrals the authors present what appears to be an easy verification of (5) and follow it by the interesting generalization

$$\prod_a^b e^{(A(\tau)+B(\tau))d\tau} = \prod_a^b e^{A(\tau)d\tau}e^{B(\tau)d\tau} \tag{6}$$

where $A(\cdot)$ and $B(\cdot)$ are continuous. In the process of deriving (5) however an equation, 3.6 (p. 135), was introduced which, while conveniently simplifying matters, is nonetheless false. (ii) The use of product integrals in proving the sufficiency (easy) half of the Hille-Yosida theorem seems superfluous.

Product integration constitutes volume 10 in the *Encyclopedia of mathematics and its applications*, and is heralded by editor Gian-Carlo Rota as "the first survey of the product integral since the turn of the century." Indeed, in the historical notes that end each chapter, and in a brief concluding chapter, *Complements*; *other work and further results on product integration*, Dollard and Friedman summarize results of other authors and convey a sense of historical development. In addition, an extensive and valuable annotated bibliography on the general subject of product integration is included. As well, an appendix prepared by P. R. Masani entitled *The place of multiplicative integration in modern analysis*, serves to indicate the relationship of product integration to the areas of Lie groups, Lie algebras and parallel transport theory. Nonetheless, even considering the aforementioned inclusions, *Product integration* remains little more than a record of the Dollard-Friedman approach to the product integral, as originally set forth in a series of six joint papers running from 1977 to 1979. Incidentally, I was surprised and disappointed that one of the authors' most significant results, Theorem 12 of *On strong product integration* (Journal of Functional Analysis, vol. 28), was not included. This theorem, in which equation (1) is solved when the operators $A(t)$ are generators of contraction semigroups on a Banach space, notably simplifies and generalizes previous results of Kato and Yosida and would have fit nicely into Chapter 3.

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Nonnegative matrices in the mathematical sciences, by Abraham Berman and Robert J. Plemmons, Academic Press, New York, 1979, xviii + 316 pp., $32.00.

Matrices with nonnegative entries occupy a very special niche in matrix theory, because of their natural importance in a wide variety of applications and because of a long list of very aesthetic mathematical properties, which, among other things, establish their role as a natural generalization of nonnegative real numbers (along side positive semi-definite matrices). Important properties are still being discovered, and, typical of much of modern research in matrix theory, the work often involves an attractive marriage of algebra, analysis, combinatorics, and geometry. It is neither primarily linear nor primarily algebraic. In addition, the properties and applications of nonnegative matrices inspire an array of generalizations both inside and outside of the finite dimensional setting.

The most fundamental facts about nonnegative matrices, established by Perron [**5**] and Frobenius [**1**] about 70 years ago, are mathematically difficult enough that, despite their extreme importance, they do not generally find their way into even advanced undergraduate matrix theory courses. (This is in part due to the fact that a rigorous author considering a chapter in a book for such a course faces a difficult choice between a long drawn out proof or the use of external tools—or no chapter.) Since there are regrettably few graduate courses in matrix theory, the number of serious expository treatments of the subject is sadly limited (to a few survey papers and a few brief book chapters, all showing signs of age, and Seneta's 1973 book [**6**] which is highly specialized). The standard sources until now have been [**2**, **7**, **8**, **9**], the most recent of which is nearly 20 years old. This is doubly unfortunate because there is a significant number of applied topics whose fundamental features are essentially obvious, given just a knowledge of the more basic theory of nonnegative matrices (such as elementary Markov processes, input-output analysis, parts of stability analysis and iterative methods, two-person zero-sum game theory and linear programming). Thus, it is fair to say that a comprehensive book on the subject was long overdue.

The book under review satisfies a good portion of the existing need, and the authors have done the mathematical and applied community a service in preparing what will be a standard reference for several years. But like any