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Polylogarithms and associated functions, by Leonard Lewin, North-Holland, Amsterdam, 1981, xvii + 359 pp., $54.95.

Dilogarithms and associated functions [**5**] was the first version of this book. The author wrote in the preface how he became interested in the subject. He recalled "glancing through the pages of Edwards' *Calculus*—a fascinating book if ever there was—when my eye caught a paragraph at the foot of the page recounting some formulae of Landen's", and then went on to mention one

$$\int_0^{(\sqrt{5}-1)/2} \log(1-x)\,dx/x = \log^2\left(\frac{\sqrt{5}-1}{2}\right) - \pi^2/10.$$

How could anyone take such a book seriously? In fact I did not see the book when it was published in 1958, but ran across it about ten years later. It seemed far too special and old fashioned, even for someone as old fashioned as I am. However, times change, and the dilogarithm has started to occur in some very fashionable places. In the notes on Thurston's work [**11**] there is a chapter on computations of volumes (by J. W. Milnor) that uses the dilogarithm. Spencer Bloch has used the dilogarithm in work on K-theory [**2**, **3**]. In the Foreword to the present book, Alf van der Poorten pointed out how some of the identities that arose while trying to understand Apéry's proof of the irrationality of $\zeta(3)$ and $\zeta(2)$ are contained in [**5**]. So what is the dilogarithm and why should there be a book on it and related functions?

The dilogarithm is the function given by

$$Li_2(z) = \sum_{n=1}^{\infty} z^n/n^2, \qquad |z| \leq 1,$$

and extended by analytic continuation. This can be effected by the integral representation

$$Li_2(z) = -\int_0^z \log(1-t)\,dt/t.$$

The first work on it seems to have been done by Landen and published in 1760. Independently Euler studied it, and they obtained results such as

$$Li_2(z) + Li_2(1-z) = \pi^2/6 - \log z \log(1-z)$$

and

$$Li_2(x) + Li_2(x/(x-1)) = -\tfrac{1}{2}\log^2(1-x), \qquad x < 1.$$

The first book on this function was by Spence [**8**] in 1809. It is a surprisingly interesting book, and if Spence had had a university position and lived longer he might have had a very good effect on mathematics in the United Kingdom, [**9**, **10**]. Up to this stage the interest in the dilogarithm was formal, i.e. it was a natural function to consider, but no one had any idea if it would be particularly useful, or if it had a natural setting. Euler considered other functions and differential equations they satisfy, and finally discovered the most important of them, the hypergeometric function and the hypergeometric differential equation. From the view point of hypergeometric functions the two functional equations above are not surprising. The second was extended by Pfaff (in 1797) to

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1(a,c-b;c;z/(z-1)). $$

This is a very important result, and as easy to prove as the above result of Landen, which is much less important.

Abel saw some results on the dilogarithm in one of Legendre's books, and then rediscovered some of Spence's work. His work was only published in the second edition of his *Oeuvres complètes*, [**1**], but this was much more widely available than Spence's book [**8**] or Spence's collected works [**9**], so the usual references are to Abel. The formula of Spence and Abel,

$$ \text{(A)} \quad \begin{aligned} &Li_2(xy/(1-x)(1-y)) = Li_2(x/(1-y)) + Li_2(y/(1-x)) \\ &\quad -Li_2(x) - Li_2(y) - \log(1-x)\log(1-y), \qquad x, y < 1, \end{aligned} $$

was the first indication that this function might arise in a natural setting. It seems to be impossible to extend (A) to hypergeometric functions. Here is a brief argument for nonspecialists to support this contention. Hypergeometric functions arise in many natural settings; as spherical functions on two point homogeneous spaces, compact and noncompact, and connected and discrete; as solutions to algebraic differential equations; and in the study of orthogonal polynomials on the real line and on the unit circle, to name just a few. Each of these settings is responsible for a number of important formulas or other facts. If a special case or limiting case of the hypergeometric function has some formulas that do not extend to the general case, then there is usually some important reason why this is so, and this is often a reflection of a natural setting for the functions. Since (A) does not seem to extend, there is probably a better reason for its existence than the simple fact that it can be proven rather easily. The first natural setting for dilogarithms was found by Lobachevsky [**6**]. Actually he considered a function equivalent to one introduced earlier by Clausen [**4**],

$$ \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2} = -\int_0^{\theta} \log\left[2\sin\frac{t}{2}\right] dt = \operatorname{Im} Li_2(e^{i\theta}). $$

He computed volumes of tetrahedrons in hyperbolic space using this function. Functional equations similar to (A) arise from decomposing the figure into subsets and finding the volume of each piece separately. Schläfli [7] used $\operatorname{Re} Li_2(re^{i\theta})$ to compute volumes in elliptic space. There was only one brief mention of this type of result in [**5**], but many more references are given in the present book. The geometric setting was not used to prove any of these identities, which is not surprising since the author is an electrical engineer and his interests center on the formulas themselves and physical applications. However he has done a nice job of surveying the literature and included many very fascinating formulas. Anyone who appreciates beautiful formulas should become familiar with this book. They should also learn something about hypergeometric functions, because many of the results on dilogarithms, polylogarithms, and the related functions of Clausen and the inverse tangent integral are really special cases of results about generalized hypergeometric series. A generalized hypergeometric series is just Σa_n with a_{n+1}/a_n a rational function of n. This advice is especially directed to the people who have made their first acquaintance with some of these functions in applications in areas where hypergeometric functions have not regularly occurred. They should recall that Euler worked for a few decades trying to find the right class of functions that would have nice integral and series representations, and a differential equation with a rich structure, and that after he found the hypergeometric function, Gauss, Kummer, Jacobi, Riemann and many others studied it extensively. They knew what they were doing, and discovered many important facts and methods that can be used to study related functions.

There is a historical curiosity. L. J. Rogers is best remembered because S. Ramanujan rediscovered two q-series identities of his. Rogers and Ramanujan both worked on the dilogarithm as well. The surprise is that generalizations of their q-series identities due to G. Andrews have led to new results on the dilogarithm. This was discovered by B. Richmond and G. Szekeres. Rogers and Ramanujan would have been pleasantly surprised by this.

There are enough additions to [**5**] so that libraries should have the present book even if they have [**5**]. The present book is easier to read since the print is slightly larger. Most of the formulas in the two books have the same numbers. This is not true in Chapter 5, where a new section has been added near the end. Formulas (5.119)–(5.126) have been added, so 8 has to be subtracted from the remaining numbers to obtain the corresponding numbers in [**5**]. Similarly, in Chapter 1, §1.9.5 has been rewritten, formulas (1.87)–(1.110) are new, and the old formulas (1.87)–(1.89) become (1.111)–(1.113). The remaining additions that have numbered formulas occur at the end of chapters.

My one complaint to the publisher is unfortunately all too common. The price is too high. One wishes North-Holland would be more confident that their books will sell to individuals rather than just to libraries. Their prices on far too many books seem to be set at a level so they will break even on library sales. This usually leads to a price that is so high that sales to individuals are small.

In closing, Barry McCoy once remarked that "Nineteenth century mathematics is alive and well and living in physics departments". This book shows that it can flourish in electrical engineering departments, and some of the recent events in mathematics show that it is not completely dead in mathematics departments.

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