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A GENERAL REGULARITY THEOREM FOR SEMILINEAR HYPERBOLIC WAVES IN ONE SPACE DIMENSION

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We study the propagation of singularities for semilinear strictly hyperbolic systems in one space dimension

$$D_i u_i = f_i(x, t, u), \quad u_i(x, 0) = u_i^{(0)}(x), \tag{1}$$

$i = 1, \ldots, m$, where the vector fields D_i and the functions f_i are C^∞. In [1] we presented a simple example which shows that the propagation of singularities for (1), is, in general, not the same as for the special case where the f_i are linear in u. In [2] we used elementary methods to study this problem when the initial data are piecewise smooth. We report here on results which permit general initial data $u^{(0)} \in H^s_{\text{loc}}$, $s > 1/2$.

The propagation of singularities for (1) is governed by two general principles:

TREE LAW. When two or more singularity bearing characteristics intersect, the point of intersection becomes a source of singularities travelling, in general, in all forward characteristic directions from the point.

SUM LAW. When H^s singularities collide with H^τ singularities, the new singularities produced will have strength $H^{s+\tau}$ (this must be interpreted in a suitable microlocal sense).

Let $S^{(0)}$ denote the union of forward characteristics from the singular support of the initial data. Let $S^{(1)}$ denote the union of forward characteristics from points of intersection in $S^{(0)}$. Continue defining $S^{(l)}$ in this way and set $S =$ closure $\bigcup^\infty S^{(l)}$. The first general principle indicates that, in general, S will be the singular support of u. The second general principle predicts the regularity of u at various points in S.

EXAMPLE 1. Let $m = 3$, $D_1 = \partial/\partial t, + \partial/\partial x$, $D_2 = \partial/\partial t - \partial/\partial x$, $D_3 = \partial/\partial t$ and suppose that $u^{(0)}$ is H^s on $[a, b]$ and C^∞ outside $[a, b]$. Then the two general principles predict the regularity depicted in Figure 1. If the f_i are linear, then u will be C^∞ in the regions labelled $H^{2s}, H^{3s}, \ldots$.

EXAMPLE 2. Suppose that $m = 4$ and that there are two rightward moving and two leftward moving characteristics. Suppose that the data is C^∞ except at two points x_1 and x_2.

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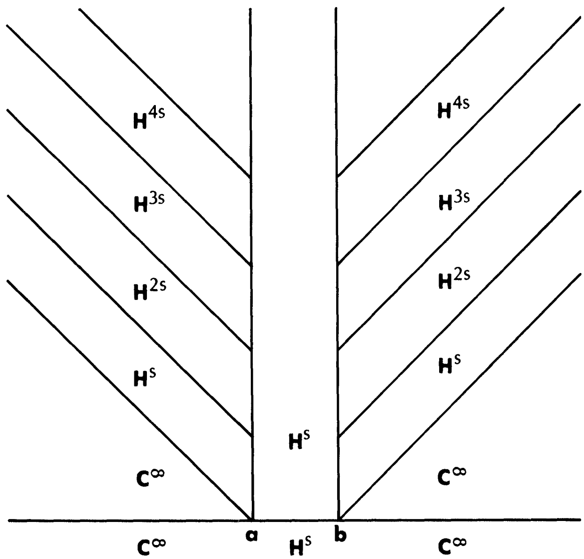

FIGURE 1. Regularity in Example 1

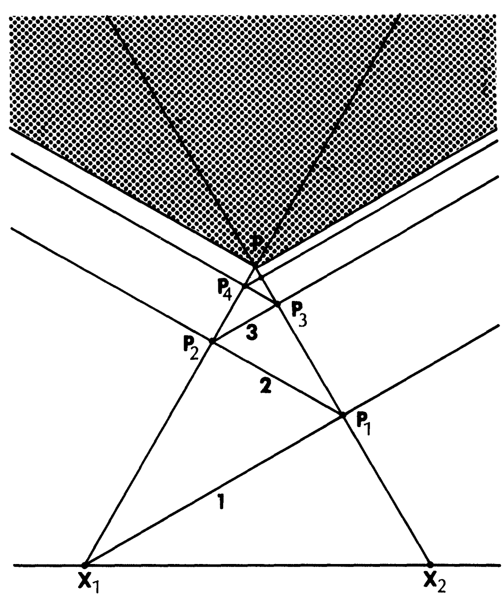

FIGURE 2

The fast rightward characteristic from x_1 (labelled 1 in Figure 2) intersects the slow leftward characteristic from x_2 at P_1. A new fast leftward singularity will be produced (labelled 2) and at P_2 a new fast rightward singularity will be produced. In this way a sequence of intersection points $P_1, P_2, P_3, \ldots$ in S will be produced such that $P_n \longrightarrow P$. It can be shown that whenever S contains such a limit point P, it will also contain the entire solid forward cone from P (shaded). Thus initial singular support of only two points can produce a solid forward cone in the singular support of the solution if $m \geqslant 4$.

In order to prove theorems about these phenomena, we develope the harmonic analysis of a family of spaces of distributions appropriate to the problem. The usual spaces defined by decay of the Fourier transform in cones do not provide fine enough tools because one needs not only the behavior on Char D_i and off Char D_i but also the behavior up to Char D_i. Further, it is important that the spaces be invariant under composition with C^∞ maps so that $f(x, t, u)$ has the same regularity as u.

Define

$$(H^s)^k_{\partial/\partial x_1}(\mathbf{R}^2) = \{u \in S'(\mathbf{R}^2) |\ (1 + \xi^2)^{s/2}(1 + \xi_1^2)^{k/2}u(\xi_1, \xi_2) \in L^2(\mathbf{R}^2)\}.$$

If $x \in \mathbf{R}^2$ and D is a C^∞ vector field, then by change of variables one can define local spaces $(H^s)^k_D(p)$ in a natural way. If k is a nonnegative integer, this is just the set of u such that $u, Du, \ldots, D^k u$ are all in H^s locally at p. In **[1]** we showed that $\bigcap_k (H^s)^k_D(p)$ is the right space for describing the regularity of u at points p where there are singularities moving in only *one* direction $D(p)$. To describe singularities moving in m directions at p we define the following spaces: Let $r_i > ½$, $i = 1, \ldots, m$, be given, $\rho = \min\{r_i + r_j,\ i \neq j\}$. Then

$$A(\vec{r}; p) \equiv \bigoplus_{\{i | r_i < \rho\}} (H^{r_i})^{\rho - r_i}_{D_i(p)}(p).$$

This is just the set of u which can be written $u = \Sigma^m\ u_i$ with $u_i \in (H^{r_i})^{\rho - r_i}_{D_i}(p)$.

THEOREM 1. *$A(\vec{r}; p)$ is an algebra invariant under C^∞ maps.*

One thinks of $A(\vec{r}; p)$ as the regularity corresponding to the phrase "singularities of strength r_i travelling in direction $D_i(p)$ at p." This interpretation suggests the following rule for predicting the regularity of u from that of its Cauchy data.

Let $\Gamma^i(q, p)$ denote the ith characteristic between q and p. Let (c, d) be an interval on which initial data $u^0 \in H^s_{\text{loc}}$, $s > ½$, are given and let R be the set of points whose initial interval of dependence is contained in (c, d). Suppose τ: $(c, d) \longrightarrow (½, \infty)$ is a function such that $p_0 \in (c, d)$ implies $u^0 \in H^{\tau(p_0)}(p_0)$.

For any p on the ith forward characteristic from p_0 we define $\sigma_i^{(0)}(p) = \tau(p_0)$. Inductively, we define,

$$\sigma_i^{(l)}(p) = \inf_{q \in \Gamma^i(p_0, p)} \left\{ \sigma_i^{(l-1)}(q), \min_{\substack{i,j,k \\ \text{distinct}}} (\sigma_j^{(l-1)}(q) + \sigma_k^{(l-1)}(q)) \right\} \tag{2}$$

and set

$$\sigma_i(p) \equiv \lim_{l \to \infty} \sigma_i^{(l)}(p),$$

$\vec{\sigma}(\cdot) = (\sigma_1(\cdot), \ldots, \sigma_m(\cdot))$ is called an index of regularity for u.

THEOREM 2. *Let $u \in L^\infty(R)$ be a distribution solution of* (1) *in R with initial data $u^0 \in H^s_{\text{loc}}$, $s > 1/2$. Let $\vec{\sigma}$ be an index of regularity for u. Then $u \in A(\vec{\sigma}(p); p)$ for each p in R.*

The main ingredients in the proof are refinements of the harmonic analysis indicated in Theorem 1 and local nonlinear elliptic and hyperbolic regularity theorems adapted to the algebras $A(\vec{r}; p)$.

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