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ORTHOGONAL TRANSFORMATIONS FOR WHICH TOPOLOGICAL EQUIVALENCE IMPLIES LINEAR EQUIVALENCE

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Let $R_1, R_2 \in O(n)$, the group of orthogonal transformations of $\mathbf{R}^n$. We say R_1 and R_2 are *topologically* (resp. *linearly*) *equivalent* if there is a homeomorphism (resp. linear automorphism) f: $\mathbf{R}^n \longrightarrow \mathbf{R}^n$ such that

(1) $$f^{-1}R_1f = R_2: \mathbf{R}^n \longrightarrow \mathbf{R}^n, \qquad f(0) = 0.$$

(Of course, linear equivalence of R_1 with R_2 is the same as equality of the respective sets of complex eigenvalues.) The *order* of an orthogonal transformation is its order as an element of $O(n)$. The purpose of this note is to announce and discuss the proof of the following result [**HP**].

THEOREM A. *Let $R_1, R_2 \in O(n)$ have order $k = l2^m$, where l is odd and $m \geqslant 0$. Suppose that*

(a) *R_1 and R_2 are topologically equivalent, and*

(b) *each eigenvalue of R_1^l and R_2^l is either* 1 *or a primitive* 2^mth *root of unity. Then R_1 and R_2 are linearly equivalent.*

If G is a group and $\rho_1, \rho_2: G \longrightarrow O(n)$ are orthogonal representations, we say ρ_1 and ρ_2 are *topologically* (resp. *linearly*) *equivalent* if there is a homeomorphism (resp. linear automorphism) f: $\mathbf{R}^n \longrightarrow \mathbf{R}^n$, $f(0) = 0$, such that

(2) $$f\rho_1(g)(x) = \rho_2(g)f(x),$$

for all $x \in \mathbf{R}^n$, $g \in G$. Here is an equivalent statement of Theorem A giving a more geometric description of its condition (b).

THEOREM B. *Let $\rho_1, \rho_2: G \longrightarrow O(n)$ be orthogonal representations of the finite group G such that $\rho_1|H$ and $\rho_2|H$ define semi-free actions of H on $\mathbf{R}^n$ for each cyclic* 2*-subgroup H of G. If ρ_1 and ρ_2 are topologically equivalent, then they are linearly equivalent.*

Returning to Theorem A, note that if k is odd, condition (b) may be omitted; in this case the result has been proved independently, using rather different methods, by Madsen and Rothenberg [**MR**]. If k is an odd prime power,

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Theorem A had been proved in **[Sc]** and, if $k \leqslant 6$, in **[KR]** where the more general question of linear versus topological equivalence of arbitrary linear endomorphisms of $\mathbf{R}^n$ was studied. (In fact, the general question was reduced to the special case of orthogonal transformations of finite order in **[KR]**.)

Unless $k = 4$, our result is the best possible, in the following sense. The remarkable results of **[CS1]** include for each $k = l2^m$, where $m \geqslant 2$, examples of topologically equivalent orthogonal transformations R_1 and R_2 of order k, where R_1^l and R_2^l each have eigenvalues of any prescribed order 2^j, $0 \leqslant j \leqslant m$, with at least one where $1 < j < m$, and where R_1 and R_2 are not linearly equivalent.

Roughly stated, the proof of Theorem A has two parts. First, the theory of Anderson and Hsiang **[AH1–3]**, which describes the obstructions to making f piecewise-linear (p.l), is applied wit.ı the consequence that in (1) above, R_1 and R_2 may be assumed to have no eigenvalues equal to 1 and f may be assumed p.l. on $\mathbf{R}^n - \{0\}$. Now add a point at infinity to the range of f, discard the origin (getting $\mathbf{R}^n$ again), and take the union of this with the domain of f, identifying corresponding points under $f|\mathbf{R}^n - \{0\}$. The result is p.l. homeomorphic to the n-sphere S^n, and R_1 and R_2 conspire to define a periodic p.l. map $R: S^n \longrightarrow S^n$ of period k.

If G denotes the cyclic group of order k, then we have constructed a p.l. G-action $G \times S^n \longrightarrow S^n$ with exactly two points x_1, x_2 fixed by G, near which G is actually acting smoothly, with its generator R inducing R_i on the tangent space to x_i. Now if G were acting everywhere smoothly on S^n, then the Atiyah-Singer G-signature formula (ASGSF) might be used to show the eigenvalues of R_1 and R_2 were the same: this general line of argument seems first to have been used by Atiyah, Bott and Milnor **[AB**, 7.15, 7.27**]**.

On the other hand, the topological equivalence of linearly inequivalent $R_i \in O(9)$ constructed in **[CS2]** was likewise p.l. on $\mathbf{R}^9 - \{0\}$. The essential difference is that in this case the ASGSF gives no information about the eigenvalues at isolated fixed points because of the presence of (-1)-eigenvalues (the Euler class of the (-1)-eigenbundle must vanish in **[AS**, 6.12**]**). Moreover, condition (b) in Theorem A avoids (-1)-eigenvalues on all powers of the R_i, unless that power has order $2l$, l odd.

Thus, in the presence of a p.l. ASGSF, we can make the argument of **[AB**, 7.27**]** work to give Theorem A. To get such a result, we first define a bordism theory of p.l. G-actions on closed p.l. manifolds, requiring (in place of the slice and tube theorems in the differentiable case) that G preserve p.l. block bundles around orbit types. The resultant bordism groups are "computable" in a way similar to that exposed in **[CF]** (again the smooth analogue). Given such a p.l. G-action $G \times M \longrightarrow M$, one defines the G-signature representation as in **[AS**, §6**]**.

We then show by bordism computations that the trace of this representation on a generator g of G depends only on the fixed point set of g and its equivariant normal (block) bundle, and that this dependency is sufficient to detect the eigenvalues at the fixed points for the p.l. G-action on S^n constructed above.

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