A TORELLI THEOREM
FOR SIMPLY CONNECTED ELLIPTIC SURFACES
WITH A SECTION AND \( p_g \geq 2 \)

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0. The purpose of this note is to announce some recent results on the
period mapping for simply connected elliptic surfaces with a section and \( p_g \geq 2 \).
Among other things, we are able to prove the period mapping for such surfaces
has degree one, that is to say, it is generically injective. In the terminology of
\[ \text{[C, G]}, \]
this is called the weak global Torelli theorem. In order to make a pre­
cise statement of our results, we must introduce some notation.

1. Let \( V \) be a surface with at most rational double points, \( V \to \mathbb{P}_1 \) its
minimal resolution; let \( \psi: V \to \mathbb{P}_1 \) be a proper analytic map, whose generic
fibre is a smooth elliptic curve. Let \( C_u = \psi^{-1}(u), \psi = \psi \circ \sigma, \) and \( \overline{C_u} = \overline{\psi^{-1}(u)} \).
Assume \( 3s \geq s: \mathbb{P}_1 \to V \) is a section of \( V \to \mathbb{P}_1 \), and that \( V \) is smooth along
\( \sigma(\mathbb{P}_1) \); let \( \overline{s} \) be the corresponding section of \( \overline{V} \to \mathbb{P}_1 \). Now assume \( p_g(V) = n \geq 1, \) and \( \overline{V} \) is a minimal surface; \( \overline{s} \cdot \overline{C_u} = 1, \overline{s}^2 = -(n + 1), \overline{C_u}^2 = 0. \) \( \overline{V} \to \mathbb{P}_1 \) is called an elliptic fibration with a section (see [K]). We also require that
\( \sigma(D) = \) a point for all irreducible curves \( D \subset \overline{V}, \) with \( \overline{s} \cdot D = 0 = \overline{C_u} \cdot D. \)

Let \( H^2(\overline{V}; \mathbb{Z})_0 = (Zc^1(\overline{C_u}) + \mathbb{Z} \cdot c_1(\overline{s}))^1, \) where \( c_1(D) \) denotes the 1st
Chern class of a divisor; this is an orthogonal direct summand of \( H^2(\overline{V}; \mathbb{Z}) \) and
hence must be a unimodular lattice. It is uniquely characterized by the integer \( n; \)
fix a copy and call it \( H; \) set \( \Lambda = H \oplus (Zc + Zs), \) where \( c \cdot H = s \cdot H = 0, \) \( c^2 = 0, \)
\( s \cdot c = 1, s^2 = -(n + 1). \)

Let \( \varphi: H^2(\overline{V}; \mathbb{Z}) \to \Lambda \) be an isomorphism of unimodular lattices \( \varphi(c_1(\overline{C_u})) = c, \varphi(c_1(\overline{s})) = s, \varphi(K_{\overline{V}}) = (n - 1) \cdot c \) where \( K_{\overline{V}} \) is the canonical divisor,
and hence \( \varphi(H^2(\overline{V}; \mathbb{Z})_0) = H. \) Following [P-S, Sâf], we will call \( \varphi \) a marking,
and the pair \( (V, \varphi) \) a marked surface of type \( (H, \Lambda, c, s). \) Let \( L_V \subset H^2(\overline{V}; \mathbb{Z})_0 \)
be the euclidean sublattice generated by all elements of the form \( c_1(D), D > 0, \)
\( \sigma(D) = \) a point; and let \( \overline{H}_V = (L_V)^1; L_V \) is negative definite. For any euclidean
lattice \( E, \) we let \( O(E) \) be the orthogonal group of \( E. \) \( \) This is a linear algebraic group
defined over \( Q; \) let \( O(E)_Z = \{ g \in O(E)_K | gE = E \}. \) For any \( \alpha \in E, \alpha^2 = -2, \) let

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1 The pairing on \( E \) may be indefinite.
\( S_\alpha \in \mathcal{O}(E)_\mathbb{Z} \) be the reflection defined by \( S_\alpha(\beta) = \beta + (\alpha \cdot \beta)\alpha, \forall \beta \in E; \) and let \( \mathcal{R}(E) \subset \mathcal{O}(E)_\mathbb{Z} \) be the subgroup generated by the reflections.

Let \((V_i, \varphi_i), i = 1, 2,\) be two marked surfaces, \( f: V_1 \to V_2 \) an isomorphism, \( \tilde{f} \) its unique lift to \( \tilde{V}_1 \to \tilde{V}_2. \) Assume that \( \exists g \in \mathcal{R}(L), \exists \varphi_1 \circ \tilde{f}^* = \tilde{g} \circ \varphi_2, \) where \( \tilde{g} \) is the obvious extension of \( g \) to \( \Lambda; \) then we say that the marked surfaces are equivalent, \( (V_1, \varphi_1) \sim (V_2, \varphi_2). \)

Let \( J = J(H, \Lambda, c, s) \) be the set of marked surfaces of type \((H, \Lambda, c, s),\)

modulo this equivalence. Let \( \Gamma = \mathcal{O}(H)_\mathbb{Z}, \) we define an action of \( \Gamma \) acts properly discontinuously on \( J, \) and \( J/\Gamma = \mathcal{M} \) is irreducible. \(^2\) Moreover, if \( n \geq 2 \) then for marked surfaces \((V_i, \varphi_i), i = 1, 2,\) \( 3f: V_1 \to V_2, \exists f \) is a birational isomorphism with \( f(s_1) = s_2 \) iff \( \exists \gamma \in \Gamma, \) such that \( \gamma \cdot (V_1, \varphi_1) \sim (V_2, \varphi_2). \)

2. Let \( \text{Gr}(n, H_C) \) be the Grassmannian of \( n \)-planes in \( H_C; \xi \in \text{Gr}(n, H_C) \) corresponding to the \( n \)-plane \( F(\xi) \subset H_C. \) Following Griffiths \([G], \) let \( \tilde{D}(H) = \{ \xi \in \text{Gr}(n, H_C) | u \cdot v = 0, \forall u, v \in F(\xi) \}; \) let \( D(H) = \{ \xi \in \tilde{D}(H) | u \cdot u > 0, \forall u \in F(\xi) - \{0\} \}. \) \( D(H) \) is an open subset of \( \tilde{D}(H), \) homogeneous under the action of \( \mathcal{O}(H)_\mathbb{R}, \) \( \Gamma \) acts properly discontinuously on \( D(H). \) Let \( z = (V, \varphi) \in J \) be a marked surface; and \( H^{2,0}(V) \subset H^2(\bar{V}, C) \) be the subspace corresponding to holomorphic \((2, 0)\) forms. Let \( \varphi: H^2(\bar{V}, C) \to \Lambda_C \) also denote the extension of \( \varphi \) to \( H^2(\bar{V}, C) \); clearly \( \varphi(H^{2,0}(V)) \subset H_C, \) since \( \varphi^{-1}(c), \varphi^{-1}(s) \) both correspond to algebraic cycles. Define \( \Phi(z) = \xi \in D(H) \) by \( \varphi(H^{2,0}(V)) = F(\xi); \) clearly \( \Phi(\gamma \cdot z) = \gamma \cdot \Phi(z). \)

By fundamental results of Griffiths, \( \Phi \) is holomorphic. Let us assume \( n \geq 2. \)

**Main Theorem.** There is a \( \Gamma \)-invariant dense open subset of \( J \) on which \( \Phi \) is injective.

**Remark.** This is equivalent to a “degree one” statement.

Following the philosophy in \([P-S, S]\), the essential idea is to reduce the problem to a “special” sublocus of \( J.\)

3. Let \( (V, \varphi) \in J, \) assume \( n \geq 2, \) then the elliptic fibration \( \psi: V \to P_1 \) is uniquely determined. Let \( C_1, \ldots, C_p \) be the singular fibres of \( \psi, \) so that \( C_u \) is smooth \( \forall u \in P'_1 = P_1 - \{a_1, \ldots, a_r\}; \) choose \( a \in P'_1, \) and \( e_1, e_2 \in H_1(C_a, \mathbb{Z}), \)
\( e_1 \cdot e_2 = 1. \) Let \( \tau_i \) be a local parameter at \( a_i, \) the loop \( \gamma: [0, 1] \to P'_1 \) defined by \( \tau_i(\gamma(t)) = e^{\exp{2\pi it}} \) defines a conjugacy class in \( \pi_1(P'_1, a), \) any member of

\(^2\) It follows from the work of Kas that \( \mathcal{M} \) is irreducible.
which will be referred to as a positive loop about $a_i$. Let $T: \pi_1(P_1, a) \to \text{SL}(2, \mathbb{Z})$ be the monodromy of $V \to P_1$, corresponding to the choice of $a$, and $e_1, e_2$. For $w \in \text{SL}(2, \mathbb{Z})$, we let $[w]$ denote its conjugacy class; $C_{a_1}$ is said to be of type $[w]$ if $T(\gamma) = [w]$ for any positive loop $\gamma$ about $a_i$ (see [M]).

We will call $(V, \phi)$ a special elliptic surface if $T(\pi_1(P_1, a)) \subset \{ \pm 1 \}$, $l_2 = (1, 0)$. Let $J_1 = \{ z \in J | z = (V, \phi) \text{ is a special elliptic surface} \}$. One can show that $J_1$ is a smooth analytic subvariety of $J$, invariant under $\Gamma$, and $J_1/\Gamma$ is irreducible. These surfaces will play a role similar to the one played by special Kummer surfaces in [P-S, Saf]; with one major exception: they will not be a dense subset.

4. Let $\Phi: J/\Gamma \to D/\Gamma$ be the map induced by $J \xrightarrow{f} D$, $D = D(H)$, $\pi: D \to D/\Gamma$ the qt. map. Assume there is a compact projective variety $Y$, divisors, $E', E'' \subset Y$, a proper surjective map $g: Y - E' - E'' \to M = J/\Gamma$; the map $f = \Phi \circ g$ extends to a proper holomorphic map $f_1, f_2: Y - E'' \to D/\Gamma$.

Moreover, assume $f^{-1}(J_1/\Gamma)$ is quasi-projective. This will all hold in our situation, due to the construction of $M$ as a quotient of an open subset of a suitable Hilbert scheme; together with the results of [G]. Let $Z \subset J_1$ be an irreducible component, then we may write

$$\Phi(J) \supset \Phi(Z) \supset \Phi(Z) - N, \quad \Phi(Z) \supset \Phi(Z) \supset \Phi(Z) - N_1$$

where $\Phi(J), \Phi(Z), N, N_1$ are all closed analytic subvarieties of $D$ (see [G]).

Consider the following statements:

(a) $(d\Phi)_z$ is injective at each $z \in Z$,

(b) $\Phi^{-1}(\Phi(Z)) \subset \bigcup_{\gamma \in \Gamma} \gamma \cdot Z$,

(c) $\Phi/Z: Z \to \Phi(Z)$ is injective,

(d) $\{ \gamma \in \Gamma | \gamma(Z) = Z \} = \{ \gamma \in \Gamma | \gamma \Phi(Z) \cap \Phi(Z) = \Phi(Z) \}$,

(e) $\Phi(Z) \subset \pi^{-1}(f_1(E' - E''))$.

The following is not difficult to prove:

**Lemma.** (a)–(e) imply the main theorem.

Let us make some remarks concerning the proofs of (a)–(d).

(a) This may be proven by using the techniques of [Kii].

(b) This is proven by the same methods used by [P-S, Saf] to characterize special Kummer surfaces, by means of their lattice of algebraic cycles.

(c) and (d) are reduced to the Torelli theorem for hyperelliptic curves, by means of the results contained in [B]. The idea of using the results in [B] may also be found in [P-S, Saf], however we use them somewhat differently. In order to understand our method of dealing with statement (e), we need to introduce some definitions.
By a *degeneration* of *elliptic fibrations* with section we will mean the following:

(i) connected analytic spaces $V, D, \dim_C V = 3, \dim_C D = 2$, proper maps, $\Psi, \rho, p$, making the following diagram commute.

\[
\begin{array}{ccc}
V & \xrightarrow{\Psi} & D \\
\downarrow{\rho} & & \downarrow{p} \\
D_e = \{ r \in C^1 | |r| < \epsilon \} & & \\
\end{array}
\]

Set $V_t = \rho^{-1}(t), \Delta_t = p^{-1}(t), \Delta$ is assumed to be connected, $\forall t \neq 0$; the generic fibre of $\Psi$ is a smooth elliptic curve.

(ii) $V_t, \forall t \neq 0$, has at most rational double points; $D - \Delta_0$ is smooth.

(iii) There is a holomorphic map $s: D \rightarrow V$ with $\Psi \circ s = \text{id}_P$, i.e. a section. We will restrict our attention to the case where $\Delta_t \cong \mathbb{P}_1, \forall t \neq 0$. Let $P$ denote the double covering of the Hirzebruch surface $\Sigma_2 \mathbb{P}_2, \mathbb{P}_1$ branched along $B \in \{3S_{\infty} + S\}$, where $S_{\infty}$ is the unique section of $\mathbb{P}_2$ with $S_{\infty}^2 = -2$, and $S \sim S_{\infty} + 2F, F \sim p_1^{-1}(u)$; we require $S_{\infty}$ to be an isolated component of $B$, thereby occurring with multiplicity $3$. Let

\[
\Psi_P: P \xrightarrow{s_P} \mathbb{P}_1,
\]

be the fibration induced by $p_2, s_P$ the section defined by $S \subset B$. Each fibre of $\Psi_P$ is a rational curve with a cusp, whose singularity has the form $z^2 = y^3$. $P$, together with its fibration, will be called a *connecting component*. The degeneration of elliptic fibrations with section

\[
\begin{array}{ccc}
V & \xrightarrow{\Psi} & D \\
\downarrow{\rho} & & \downarrow{p} \\
D_e = \{ r \in C^1 | |r| < \epsilon \} & & \\
\end{array}
\]

for which $V/D_e - \{0\}$ has trivial monodromy as a fibration of surfaces with rational double points, will be called *stable* if the following holds:

1. $\Delta_0$ has only ordinary nodes, and $p$ vanishes to multiplicity one on each component of $\Delta_0$. Set $\Delta_0 = \bigcup_{i \in I} \Gamma_i$, each $\Gamma_i \cong \mathbb{P}_1$, and $X_i = \Psi^{-1}(\Gamma_i)$. Each $X_i$ occurs with multiplicity one.

2. Each fibre of $\Psi$, when reduced, is an irreducible curve. Set $C_u = \Psi^{-1}(u), \forall u \in D; V$ is smooth along $s(D)$.

3. If the generic fibres of $X_i \rightarrow \Gamma_i$ and $X_j \rightarrow \Gamma_j$ are smooth then $\Gamma_i \cap \Gamma_j = \emptyset$.  

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(4) If the generic fibre of $X_t \to \Gamma_t$ is smooth, $A_0 - T_t \cap \Gamma_t = \{q_1, \ldots, q_l\}$, we let $\mu_i$ be the number of singular fibres of $X_t|_{\Gamma_t - \{q_1, \ldots, q_l\}}$, properly counted. If the generic fibre of $X_t \to \Gamma_t$ is singular, set $\mu_i = 0$. We require that $\Sigma \mu_i = 12(n + 1)$, where $p_g(V_{t_0}) = n$, $t_0 \neq 0$.

(5) If the generic fibre of $X_t \to \Gamma_t$ is smooth, and $p_g(X_t) = 0$ then $\Gamma_t$ meets at least two other components of $\Delta_0$; if $\Gamma_t$ meets exactly two other components, then $\mu_i > 0$.

(6) If the generic fibre of $X_k \to \Gamma_k$ is singular, then $X_k$ is a connecting component and $\Gamma_k$ meets exactly two other components of $\Delta_0$, $\Gamma_k'$, $\Gamma_k''$. Set $\{q'\} = \Gamma_k' \cap \Gamma_k$, $\{q''\} = \Gamma_k'' \cap \Gamma_k$, then the generic fibres of $X_{k'} \to \Gamma_k'$, $X_{k''} \to \Gamma_k''$ are smooth; and let $C_{q'}$ be of type $[w_1]$, $C_{q''}$ of type $[w_2]$. Let $w_i = s_i u_i$, $i = 1, 2$ be their Jordan decomposition into semisimple and unipotent elements, $s_i$ semisimple, $u_i$ unipotent. Then, $u_1 \neq 1$ iff $u_2 \neq 1$; and $[s_1^{-1}] = [s_2] \neq [1]$.

**Theorem (stable reduction).** Let $V, D$ etc. be a degeneration of elliptic fibrations with section, such that $V|_{D'_e}, D'_e = D_e - \{0\}$ has trivial monodromy as a fibration of surfaces with rational double points, then after a base extension $\varphi: D_{e_1} \to D_e, V \times \varphi D_{e_1}$ is $D_{e_1}$-birationally equivalent to a stable degeneration.

This theorem is the key element in obtaining a proof of statement (e), above.

To use it, we let the $Y$, used to dominate $M$, be obtained from a certain Hilbert scheme, $X \to Y$ the universal surface over $Y$, the fibres of $X|Y - E' - E''$ being elliptic surfaces with at most rational double points $E' - E'' = E_1 \cup \cdots \cup E_m \cup \widetilde{E}$, where $X|_{E_i}$ has only fibres which fulfill properties (1)—(6) above, and $f_i(E) \subset \bigcup_{j} f_j(E_j)$. Due to properties (1)—(6), if we let $E$ be any one of the $E_i$, 3 euclidean sublattices $H_j \subset H$, $1 \leq j \leq r$ each $H_j$ having a nonzero even number of positive eigenvalues, $(H_1 + \cdots + H_r)^{1 \leq i \neq j}$ being negative definite, $H_j \subset H_j^{1 \leq i \neq j}$; and a holomorphic map of $\widetilde{D} \to \Gamma$, $D$ inducing the inclusion $(H_1 + \cdots + H_r)^{1 \leq i \neq j} \subset H_j^{1 \leq i \neq j}$ induced by the inclusion $(H_1 + \cdots + H_r) \subset H$. Moreover, $\Phi(\widetilde{D}/\Gamma) \supset f_1(E)$.

Let $\Gamma_Z = \{y \in \Gamma|\gamma(Z) = Z\}$; $A_R(\Gamma_Z)$ its Zariski closure in $O(H)^R$; then $\Phi(Z) \subset \pi^{-1}(f_1(E))$ implies that some conjugate of $A_R(\Gamma_Z)$ must leave an orthogonal decomposition $H_R = H' + H''$, invariant, each $H', H''$ having a nonzero number of positive eigenvalues. It may be verified that this is impossible.

**References**


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3 It follows from this, and condition (6), that $\Sigma p_g(X_t) = n$. 

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