A SOLUTION TO A PROBLEM OF J. R. RINGROSE

BY DAVID R. LARSON

We announce a solution to a multiplicity problem for nests posed by J. R. Ringrose approximately twenty years ago. This also answers a question posed by R. V. Kadison and I. M. Singer, and independently by I. Gohberg and M. Krein concerning the invariant subspace lattice of a compact operator. The key to the proof is a result concerning compact perturbations of nest algebras which was recently obtained by Niels Andersen in his doctoral dissertation. The complete proof of the general result as well as of a number of related results will appear elsewhere. A proof for the special case which answers Ringrose’s original question is included herein.

Let $H$ be infinite dimensional separable Hilbert space. A nest $N$ is a family of closed subspaces of $H$ linearly ordered by inclusion. $N$ is complete if it contains $\{0\}$ and $H$ and contains the intersection and the join (closed linear span) of each subfamily. The corresponding nest algebra $\text{alg} N$ is the algebra of all operators in $L(H)$ which leave every member of $N$ invariant. The core $C_n$ is the von Neumann algebra generated by the projections on the members of $N$, and the diagonal $D_n$ is the von Neumann algebra $(\text{alg} N) \cap (\text{alg} N)^*$. $N$ is continuous if no member of $N$ has an immediate predecessor or immediate successor. Equivalently, $N$ is continuous if the core $C_n$ is a nonatomic von Neumann algebra. $N$ has multiplicity one (is multiplicity free) if $D_n$ is abelian, or equivalently, if $C_n$ is a m.a.s.a.

J. R. Ringrose posed the following question: Let $N$ be a multiplicity free nest and $T: H \rightarrow H$ a bounded invertible operator. Is the image nest $TN = \{TN: N \in N\}$ necessarily multiplicity free? Note that $T(\text{alg} N)T^{-1} = \text{alg}(TN)$, so it is natural to say that $TN$ is the similarity transform of $N$. Is multiplicity preserved under similarity? We show that the answer is no. It should be noted that a negative answer was conjectured in recent years by several mathematicians including J. Ringrose and W. Arveson.

The following key result is due to N. Andersen [1]. Let $LC$ denote the compact operators in $L(H)$.

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243

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THEOREM (ANDERSEN). If $H$ is separable, and if $N, M$ are arbitrary continuous nests in $H$, then there exists a unitary operator $U$ such that $\text{alg } N + LC = U(\text{alg } M + LC)U^* = \text{alg}(UM) + LC$.

Next, we answer Ringrose's question.

THEOREM 1. Let $N$ be a continuous nest of multiplicity 1. Then there exists a positive invertible operator $T \in L(H)$ such that $TN = \{TN: N \in \mathbb{N}\}$ fails to have multiplicity 1.

PROOF. By Andersen's theorem there exists a continuous nest $M$ not of multiplicity 1 such that $\text{alg } M + LC = \text{alg } N + LC$. Since for any algebra $A$ we have $A + LC/\text{LC} \simeq A/A \cap \text{LC}$, the algebras $\text{alg } M/\text{alg } M \cap \text{LC}$ and $\text{alg } N/\text{alg } N \cap \text{LC}$ are algebraically isomorphic. The diagonal $\mathcal{D}_M = \text{alg } M \cap (\text{alg } M)^*$ is a nonabelian von Neumann algebra so contains a nonzero partial isometry $v$ with orthogonal initial and final spaces. Let $\tilde{S} = v + v^* - vv^* - v^*v + I$ and $\tilde{P} = vv^*$. Then $\tilde{S}^2 = I$ and $\tilde{P}\tilde{S}\tilde{P} = 0$. Since $M$ is continuous $\mathcal{D}_M$ contains no compacts, so $\tilde{P}$ has infinite rank. Thus via the algebraic isomorphism between quotients it follows that $\text{alg } N/\text{alg } N \cap \text{LC}$ contains elements $\tilde{P}, \tilde{S}$ with $\tilde{P}^2 = \tilde{P} \neq 0, \tilde{S}^2 = I, \tilde{P}\tilde{S}\tilde{P} = 0$.

Let $A$ and $B$ be elements of $\text{alg } N$ whose images in the quotient are $\tilde{P}$ and $\tilde{S}$ respectively. Then $A^2 - A, B^2 - I$ and $ABA$ are contained in $\text{alg } N \cap \text{LC}$, and this is contained in the Jacobson radical $\mathcal{R}_N$ of $\text{alg } N$ since $N$ is continuous. So $B$ is invertible in $\text{alg } N$. Also, a well-known result [cf. 9, Theorem 2.3.9] states that an element of a Banach algebra which is idempotent modulo the radical is equal modulo the radical to an idempotent. So there exists an idempotent $P \in \text{alg } N$ with $P - P \in \mathcal{R}_N$. We have $P \neq 0$ since otherwise $A$ would be in $\mathcal{R}_N$ and hence $\tilde{P}$ above would be a quasinilpotent idempotent, hence 0.

We have $PBP \in \mathcal{R}_N$. Set $B_1 = B - PBP$. Then $B_1$ is also invertible in $\text{alg } N$, and $PB_1P = 0$. Now set

$$S = B_1P + PB_1^{-1}(I - P) - B_1PB_1^{-1}(I - P) + I - P.$$ 

We have $PSP = 0$, and it can be verified that $S^2 = I$. (Let $\alpha$ denote the sum of the first two terms, $\beta$ the sum of the remaining terms, and compute $\beta^2 = \beta, \alpha\beta = \beta\alpha = 0, \alpha^2 = I - \beta$.)

Let $R = I - 2P$. Then $R^2 = I, S^2 = I, RS \neq SR$. Let $G$ be the group generated by $R, S$. We have $SRS = I - 2SPS$, and $PSP = 0$, so $PSRS = P = SRSP$. Hence $R$ commutes with $SRS$ since $P$ does. It easily follows that

$$G = \{I, S, R, RS, SR, SRS, RSR, SRSR\}.$$ 

So $G$ is a finite noncommutative group contained in $\text{alg } N$. Set $T = (\Sigma_{g \in G} g^*g)^{1/2}$. Then $TGT^{-1} = \{TgT^{-1}: g \in G\}$ is a noncommutative group of
unitaries contained in the diagonal of $\text{alg}(TN)$, and thus $TN$ fails to have multiplicity 1. □

Theorem 1 serves to answer an open question concerning invariant subspace lattices of compact operators due to Kadison and Singer [5] and to Gohberg and Krein [4]. An operator is said to be hyperintransitive if its lattice of invariant subspaces contains a multiplicity one nest.

**Corollary 2.** There exists a nonhyperintransitive compact operator.

**Proof.** Let $V$ be the Volterra operator. Then $\text{Lat } V$ is a continuous multiplicity one nest. Let $N = \text{Lat } V$ and let $T$ be an invertible operator such that $TN$ does not have multiplicity one. Since $\text{Lat}(TVT^{-1}) = TN$ and since $TN$ is a maximal nest the similarity $TVT^{-1}$ is not hyperintransitive. □

**Remark.** It was known for a number of years that a negative resolution to the Ringrose problem would yield Corollary 2. I believe that this connection was first observed by J. Erdos, and it was first shown to me by W. Arveson.

We strengthen Theorem 1 as follows.

**Theorem 3.** Let $N$ be a continuous nest of multiplicity one. Then given $\epsilon > 0$ there exists a positive invertible operator $T \in L(H)$ with $T - I$ compact and $\|T - I\| < \epsilon$ such that $TN = \{TN: N \in N\}$ fails to have multiplicity one.

A nest has purely atomic core if its core is generated by its minimal projections. The following shows that similarity transforms can fail to act "absolutely continuously" on nests.

**Theorem 4.** If $N$ is a complete uncountable nest with purely atomic core there exists a positive invertible operator $T$ such that $TN$ does not have purely atomic core.

A nest $N$ is said to have the factorization property if every invertible positive operator $T$ factors $T = A^*A$ for $A \in (\text{alg } N) \cap (\text{alg } N)^{-1}$. Arveson [2] proved that nests of the "simplest type" have the factorization property. We generalize this to countable complete nests, and then show that these are the only ones with this property.

**Theorem 5.** A complete nest has the factorization property if and only if it is countable.

In contrast, if we drop the requirement that $A^{-1}$ also be in $\text{alg } N$ we obtain.

**Theorem 6.** Let $N$ be an arbitrary nest. Then every invertible positive operator $T$ factors $T = A^*A$ for $A \in \text{alg } N$, $A$ invertible in $L(H)$.

The following answers a question of J. Erdos [3].
THEOREM 7. Let $N$ be a continuous nest. Then the commutator ideal of $\text{alg } N$ is not proper.

REFERENCES