A SOLUTION TO A PROBLEM OF J. R. RINGROSE

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We announce a solution to a multiplicity problem for nests posed by J. R. Ringrose approximately twenty years ago. This also answers a question posed by R. V. Kadison and I. M. Singer, and independently by I. Gohberg and M. Krein concerning the invariant subspace lattice of a compact operator. The key to the proof is a result concerning compact perturbations of nest algebras which was recently obtained by Niels Andersen in his doctoral dissertation. The complete proof of the general result as well as of a number of related results will appear elsewhere. A proof for the special case which answers Ringrose’s original question is included herein.

Let $H$ be infinite dimensional separable Hilbert space. A nest $\mathcal{N}$ is a family of closed subspaces of $H$ linearly ordered by inclusion. $\mathcal{N}$ is complete if it contains $\{0\}$ and $H$ and contains the intersection and the join (closed linear span) of each subfamily. The corresponding nest algebra $\text{alg} \mathcal{N}$ is the algebra of all operators in $L(H)$ which leave every member of $\mathcal{N}$ invariant. The core $C_{\mathcal{N}}$ is the von Neumann algebra generated by the projections on the members of $\mathcal{N}$, and the diagonal $D_{\mathcal{N}}$ is the von Neumann algebra $(\text{alg} \mathcal{N}) \cap (\text{alg} \mathcal{N})^*$. $\mathcal{N}$ is continuous if no member of $\mathcal{N}$ has an immediate predecessor or immediate successor. Equivalently, $\mathcal{N}$ is continuous if the core $C_{\mathcal{N}}$ is a nonatomic von Neumann algebra. $\mathcal{N}$ has multiplicity one (is multiplicity free) if $D_{\mathcal{N}}$ is abelian, or equivalently, if $C_{\mathcal{N}}$ is a m.a.s.a.

J. R. Ringrose posed the following question: Let $\mathcal{N}$ be a multiplicity free nest and $T: H \to H$ a bounded invertible operator. Is the image nest $TN = \{TN: N \in \mathcal{N}\}$ necessarily multiplicity free? Note that $T(\text{alg} \mathcal{N})T^{-1} = \text{alg}(TN)$, so it is natural to say that $TN$ is the similarity transform of $\mathcal{N}$. Is multiplicity preserved under similarity? We show that the answer is no. It should be noted that a negative answer was conjectured in recent years by several mathematicians including J. Ringrose and W. Arveson.

The following key result is due to N. Andersen [1]. Let $LC$ denote the compact operators in $L(H)$.
THEOREM (ANDERSEN). If $H$ is separable, and if $N, M$ are arbitrary continuous nests in $H$, then there exists a unitary operator $U$ such that $\text{alg } N + LC = U(\text{alg } M + LC)U^* = \text{alg}(UM) + LC$.

Next, we answer Ringrose's question.

THEOREM 1. Let $N$ be a continuous nest of multiplicity 1. Then there exists a positive invertible operator $T \in L(H)$ such that $TN = \{TN : N \in N\}$ fails to have multiplicity 1.

PROOF. By Andersen's theorem there exists a continuous nest $\tilde{M}$ not of multiplicity 1 such that $\text{alg } \tilde{M} + LC = \text{alg } N + LC$. Since for any algebra $A$ we have $A + LC/LC \cong A/A \cap LC$, the algebras $\text{alg } M/\text{alg } M \cap LC$ and $\text{alg } N/\text{alg } N \cap LC$ are algebraically isomorphic. The diagonal $\text{diag}_M = \text{alg } M \cap (\text{alg } M)^*$ is a non-abelian von Neumann algebra so contains a nonzero partial isometry $v$ with orthogonal initial and final spaces. Let $\tilde{S} = v + v^* - vv^* - v^*v + I$ and $\tilde{P} = vv^*$. Then $\tilde{S}^2 = I$ and $\tilde{P}\tilde{S}\tilde{P} = 0$. Since $\tilde{M}$ is continuous $\text{diag}_M$ contains no compacts, so $\tilde{P}$ has infinite rank. Thus via the algebraic isomorphism between quotients it follows that $\text{alg } N/\text{alg } N \cap LC$ contains elements $\tilde{P}$, $\tilde{S}$ with $\tilde{P}^2 = \tilde{P} \neq 0$, $\tilde{S}^2 = I$, $\tilde{P}\tilde{S}\tilde{P} = 0$.

Let $A$ and $B$ be elements of alg $N$ whose images in the quotient are $\tilde{P}$ and $\tilde{S}$ respectively. Then $A^2 - A$, $B^2 - I$ and $ABA$ are contained in alg $N \cap LC$, and this is contained in the Jacobson radical $R_N$ of alg $N$ since $N$ is continuous. So $B$ is invertible in alg $N$. Also, a well-known result [cf. 9, Theorem 2.3.9] states that an element of a Banach algebra which is idempotent modulo the radical is equal modulo the radical to an idempotent. So there exists an idempotent $P \in \text{alg } N$ with $A - P \in R_N$. We have $P \neq 0$ since otherwise $A$ would be in $R_N$ and hence $\tilde{P}$ above would be a quasinilpotent idempotent, hence $0$.

We have $PB^\perp P \in R_N$. Set $B_1 = B - PB^\perp P$. Then $B_1$ is also invertible in alg $N$, and $PB_1^2 P = 0$. Now set

$$S = B_1 P + PB_1^2 (I - P) - B_1 PB_1^{-1} (I - P) + I - P.$$ 

We have $PSP = 0$, and it can be verified that $S^2 = I$. (Let $\alpha$ denote the sum of the first two terms, $\beta$ the sum of the remaining terms, and compute $\beta^2 = \beta$, $\alpha \beta = \beta \alpha = 0$, $\alpha^2 = I - \beta$.)

Let $R = I - 2P$. Then $R^2 = I$, $S^2 = I$, $RS \neq SR$. Let $G$ be the group generated by $R, S$. We have $SRS = I - 2SP$, and $PSR = 0$, so $PSRS = P = SRSP$. Hence $R$ commutes with $SRS$ since $P$ does. It easily follows that

$$G = \{I, S, R, RS, SR, SRS, RSR, SRSP\}.$$ 

So $G$ is a finite noncommutative group contained in alg $N$. Set $T = (\Sigma_{g \in G} g^*g)^{1/2}$. Then $TGT^{-1} = \{TgT^{-1} : g \in G\}$ is a noncommutative group of
unitaries contained in the diagonal of \( \text{alg}(TN) \), and thus \( TN \) fails to have multiplicity 1. \( \square \)

Theorem 1 serves to answer an open question concerning invariant subspace lattices of compact operators due to Kadison and Singer [5] and to Gohberg and Krein [4]. An operator is said to be hyperintransitive if its lattice of invariant subspaces contains a multiplicity one nest.

**Corollary 2.** There exists a nonhyperintransitive compact operator.

**Proof.** Let \( V \) be the Volterra operator. Then \( \text{Lat} \ V \) is a continuous multiplicity one nest. Let \( N = \text{Lat} \ V \) and let \( T \) be an invertible operator such that \( TN \) does not have multiplicity one. Since \( \text{Lat}(TVT^{-1}) = TN \) and since \( TN \) is a maximal nest the similarity \( TVT^{-1} \) is not hyperintransitive. \( \square \)

**Remark.** It was known for a number of years that a negative resolution to the Ringrose problem would yield Corollary 2. I believe that this connection was first observed by J. Erdos, and it was first shown to me by W. Arveson.

We strengthen Theorem 1 as follows.

**Theorem 3.** Let \( N \) be a continuous nest of multiplicity one. Then given \( \epsilon > 0 \) there exists a positive invertible operator \( T \in L(H) \) with \( T - I \) compact and \( \|T - I\| < \epsilon \) such that \( TN = \{TN: N \in N\} \) fails to have multiplicity one.

A nest has purely atomic core if its core is generated by its minimal projections. The following shows that similarity transforms can fail to act "absolutely continuously" on nests.

**Theorem 4.** If \( N \) is a complete uncountable nest with purely atomic core there exists a positive invertible operator \( T \) such that \( TN \) does not have purely atomic core.

A nest \( N \) is said to have the factorization property if every invertible positive operator \( T \) factors \( T = A^*A \) for \( A \in (\text{alg } N) \cap (\text{alg } N)^{-1} \). Arveson [2] proved that nests of the "simplest type" have the factorization property. We generalize this to countable complete nests, and then show that these are the only ones with this property.

**Theorem 5.** A complete nest has the factorization property if and only if it is countable.

In contrast, if we drop the requirement that \( A^{-1} \) also be in \( \text{alg } N \) we obtain.

**Theorem 6.** Let \( N \) be an arbitrary nest. Then every invertible positive operator \( T \) factors \( T = A^*A \) for \( A \in \text{alg } N, A \) invertible in \( L(H) \).

The following answers a question of J. Erdos [3].
THEOREM 7. Let $\mathcal{N}$ be a continuous nest. Then the commutator ideal of $\text{alg } \mathcal{N}$ is not proper.

REFERENCES