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UNIPOTENT AND PROUNIPOTENT GROUPS: COHOMOLOGY AND PRESENTATIONS

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A pro-affine algebraic group G, over the field k (which we always take to be algebraically closed of characteristic zero) is an inverse limit of affine algebraic groups [**3**]. If the algebraic groups in the inverse system are unipotent, we call G *prounipotent*. Pro-affine algebraic groups arise naturally in the theory of finite-dimensional k-representations of discrete and analytic groups [**3**, **4**, **9**] and prounipotent groups arise naturally as the prounipotent radicals of pro-affine groups. Our interest in prounipotents is motivated by possible applications to finite-dimensional representation theory.

The extension of the category of unipotent groups to that of prounipotents makes possible "combinatorial group theory" (free groups and presentations):

If X is a set, there is a prounipotent group $F(X)$ containing X such that for every prounipotent group H and function $f\colon X \longrightarrow H$ with $\mathrm{Card}\{X - f^{-1}(L)\}$ finite for every closed subgroup L of finite codimension in H there is a unique homomorphism $\bar{f}\colon F(X) \longrightarrow H$ extending f [**5**, 2.1]. Every prounipotent group G is a homomorphic image of a free prounipotent group F so there is an exact sequence $(*)$ $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$. We can choose $(*)$ with $R \subseteq (F, F)$ and in this case we call $(*)$ a *proper presentation* of G. If $F = F(X)$ in $(*)$, we call X generators for G and we call generators of R, as a prounipotent normal subgroup of F, relations for G.

As for pro-p groups [**11**], the numbers of generators and relations for G have a cohomological interpretation. Cohomology here is in the category of polynomial representations as in [**2**]. There is a unique simple in this category (the one-dimensional trivial module k) so cohomological dimension is defined as $\mathrm{cd}(G) = \inf\{i \mid H^n(G, k) = 0, n > i\}$.

THEOREM 1 [**5**, 2.8 AND 2.9]. *The following are equivalent for prounipotent G*:

(a) *G is free,*

(b) *G is a projective group in the category of prounipotent groups,*

(c) $\mathrm{cd}(G) \leqslant 1$.

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PROPOSITION 2 [5, 1.14]. *If H is a prounipotent subgroup of the prounipotent group G then* $\mathrm{cd}(H) \leqslant \mathrm{cd}(G)$.

COROLLARY 3 [5, 2.10]. *A closed subgroup of a free prounipotent group is free.*

THEOREM 4 [5, 3.2 AND 3.11]. *Let $1 \longrightarrow R \longrightarrow F(X) \longrightarrow G \longrightarrow 1$ be a proper presentation of the prounipotent group G. Then* $d = \dim(H^1(G, k)) = \mathrm{Card}(X)$ *and* $r = \dim(H^2(G, k))$ *is the minimal number of normal generators of R as a prounipotent subgroup of F. Thus, d is the minimal number of generators and r is the minimal number of relations for G.*

The preceding results are proved similarly to the analogous results for pro-p groups. (See [11].) Special properties of prounipotents extablish

THEOREM 5 [5, 3.14]. *If G is prounipotent and* $\dim(H^n(G, k)) = 1$ *for some $n \geqslant 1$, then* $\mathrm{cd}(G) = n$.

If G is one-relator, $\dim(H^2(G, k)) = 1$ by Theorem 4 so

COROLLARY 6 [5, 3.15]. *A one-relator prounipotent group has cohomological dimension* 2.

(Corollary 6 is the prounipotent analogue of [9, 11.2, p. 633].)

When G is finite-dimensional, $\mathrm{cd}(G) = \dim(G)$, so the only one-relator G is $k \times k$. In general, there is a Golod-Shafarevich type inequality relating the numbers of generators and relations.

THEOREM 7 [7, 3.11]. *Let G, d, and r be as in Theorem 4 with $r \neq 0$ and G finite-dimensional. Then $r \geqslant d^2/4$, with strict inequality unless $G = k \times k$, when $r = 1$ and $d = 2$.*

The proof of Theorem 7 relies on the notion of a group algebra developed in [6 and 7]: The coordinate ring $k[G]$ of the prounipotent group G is a G-bimodule so that the right translations define an embedding ρ of G in the units of the G-module endomorphism ring of $k[G]$ as a left G-module. We denote $\mathrm{End}_G(k[G])$ by $k\langle\langle G\rangle\rangle$.

When G is finitely generated, $k\langle\langle G\rangle\rangle$ is like a group algebra for G (if B is a finite-dimensional associative algebra, $U_1(B)$ is the group of units of B congruent to 1 modulo the radical).

THEOREM 8 [7, 2.8]. *If G is a finitely generated prounipotent group and B a finite-dimensional associative k-algebra any polynomial representation $G \longrightarrow U_1(B)$ extends uniquely to an algebra homomorphism $k\langle\langle G\rangle\rangle \longrightarrow B$. Moreover, this property characterizes $k\langle\langle G\rangle\rangle$.*

THEOREM 9 [7, 2.10]. *Let G be a prounipotent group with a proper presentation $1 \longrightarrow R \longrightarrow F(\{x_1, \ldots, x_d\}) \longrightarrow G \longrightarrow 1$ where $\{s_1, \ldots, s_r\}$ is a minimal set of normal generators of R. Then $k\langle\langle G\rangle\rangle$ is the formal (noncommutative) power series algebra $k\langle\langle \rho(x_1) - 1, \ldots, \rho(x_d) - 1\rangle\rangle$ modulo the ideal generated by $\{\rho(s_i) - 1\}$.*

Theorem 9 is proved by first treating the case where G is free on $\{x_1, \ldots, x_d\}$ [**6**, 1.5] (so $k\langle\langle G\rangle\rangle$ is a formal power series algebra). Then the embedding $\rho: G \longrightarrow k\langle\langle G\rangle\rangle$ embeds G in the ring of formal power series. This extends (in fact, reproves) the Magnus embedding [**1**, p. 151] of the free discrete group, and provides a concrete description of the free prounipotent group on d generators as the Zariski closure of the subgroup generated by $\{1 + t_i\}$ in the group of units of constant term 1 in $k\langle\langle t_1, \ldots, t_d\rangle\rangle$. Using this description, we obtain

THEOREM 10 [**6**, 2.7]. *The associated graded Lie algebra* [**1**, p. 145] *of the lower central series of a free prounipotent group on d generators is a free k-Lie algebra on d generators.*

The proofs of the preceding theorems use a description of $k[G]$ as an ascending union of G-submodules $E_i(G)$ defined by $E_{-1}(G) = 0$ and $E_{i+1}(G)/E_i(G) = (k[G]/E_i(G))^G$. If G is finitely generated then the numbers $c_i(G) = \dim(E_i(G))$ are all finite, and we have

PROPOSITION 11 [**6**, 1.3 AND **7**, 3.12]. *Let G be prounipotent.*

(a) *G is free on d generators if and only if $c_i(G) = 1 + d + d^2 + \cdots + d^i$ for $i \geqslant 0$.*

(b) *G is finite-dimensional if and only if the series $\{c_i(G)\}$ has polynomial growth.*

Finally, we record some applications to the finite-dimensional representation theory of a discrete group Γ. We let $A(\Gamma)$ be the pro-algebraic hull of Γ [**10**, 2.2] and $R_u(\Gamma)$ the prounipotent radical of $A(\Gamma)$.

THEOREM 12 [**5**, 4.3]. *If Γ contains a free subgroup of finite index, $R_u(\Gamma)$ is a free unipotent group.*

If Γ is torsion free nilpotent, then $R_u(\Gamma)$ is finite-dimensional, and there is an embedding $\Gamma \longrightarrow R_u(\Gamma)$. (This is the Malcev embedding for which our methods provide a new proof [**6**, 5.12].) In this case we have $H^i(\Gamma, k) = H^i(R_u(\Gamma), k)$ [**7**, 3.8] so we can apply Theorem 7 to obtain an inequality relating the ranks of the first and second cohomology groups of Γ.

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