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BROWNIAN MOTION, GEOMETRY, AND GENERALIZATIONS OF PICARD'S LITTLE THEOREM

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ABSTRACT. Brownian motion is introduced as a tool in Riemannian geometry, and it is shown to be useful in the function theory of manifolds, as well as in the study of maps between manifolds. As applications, a generalization of Picard's little theorem, and a version of it for Riemann surfaces of large genus are given.

1. Picard's theorem for nonhyperbolic manifolds. Let M and N be complete Riemannian manifolds with metrics ${}^Mg$, ${}^Ng$, resp. Assume $F\colon M \longrightarrow N$ is a C^2 map. F is said to be *harmonic* [2] if its second fundamental form has trace 0. Define the tensor

$$\xi^{\alpha\beta}(x) = {}^Mg_{ij}\left[\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j}\right](x),\ x \in M.$$

Since $(\xi^{\alpha\beta}(x))$ is a symmetric matrix, its eigenvalues are nonnegative, and we may order them as follows: $\lambda_1(x) \geqslant \lambda_2(x) \geqslant \cdots \geqslant \lambda_n(x) \geqslant 0$. F is said to be *K-quasiconformal* [5] if $\lambda_1(x) \leqslant K^2\lambda_n(x)$ for all $x \in M$.

We define polar coordinates (r, θ) on N via the exponential map. There will be two restrictions on the curvature of N:

(i) *The sectional curvatures of N are bounded below by $-L^2 < 0$.*

(ii) *Each of the sectional curvatures at $(r, \theta) \in N$ determined by dr and some other tangent vector, is bounded above by $K(r)$, where $K(r)$ satisfies* (a) *for some $\epsilon > 0$, $-K(r) \sim r^{2\epsilon-2}$*; (b) *there exists a C^∞ solution $u(r)$ of the equation*

$$u''(r) = K(r)u(r), \qquad u(0) = 0, \quad u'(0) = 1,$$

and $u'(r)$ is always positive.
(Note that such a solution can always be found if $K(r)$ is negative.)

THEOREM 1. *Suppose M and N are as above with the curvature of N satisfying* (i) *and* (ii). *Then, if Brownian motion on M has trivial tail σ-field, every K-quasiconformal harmonic map $F\colon M \longrightarrow N$ is constant.*

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2. Proof of Theorem 1. The following lemmas are essential.

LEMMA 2. *Let X_t be Brownian motion on M, with X_0 chosen so that $F(X_0) = x_N \in N\backslash\{x_0\}$. Then, there is a random time change $\sigma(t)$ $(= \int_0^t ds/\lambda_1(x_s))$ and a constant $C > 0$, such that if $\rho_t = r \circ F(X_{\sigma(t)}), d\rho_t = a(X_{\sigma(t)})dB_t + b(X_{\sigma(t)})dt$, when $F(X_t) \neq x_0$, where B_t is some Brownian motion, $1/K \leqslant |a(X_{\sigma(t)})| \leqslant 1$ and $b(X_{\sigma(t)}) \geqslant C\rho_t^{\epsilon/2-1}$ if ρ_t is larger than some constant R.*

The statement uses the stochastic calculus discussed in [8].

Let τ be a stopping time for $X_{\sigma(t)}$.

LEMMA 3. *If $\rho_t^{(\tau)}$ is the distance of $F(X_{\sigma(t)})$ from $F(X_{\sigma(\tau)})$, then for $t > \tau$ and $\rho_t^{(\tau)} \neq 0$, $d\rho_t^{(\tau)} = a^{(\tau)}(X_{\sigma(t)})dB_t^{(\tau)} + b^{(\tau)}(X_{\sigma(t)})dt$, where $1/K \leqslant |a^{(\tau)}(X_{\sigma(t)})| \leqslant 1$ and $|b^{(\tau)}(X_{\sigma(t)})| \leqslant (n/2)L \coth L\rho_t^{(\tau)}$.*

The proofs of Lemmas 2 and 3 require Ito's lemma, the K-quasiconformal condition, and the Hessian comparison theorem of Greene and Wu [3]. Lemma 2 is used to determine the speed with which ρ_t goes to ∞.

LEMMA 4. $\liminf_{t\to\infty} \rho_t/t^{2/4-\epsilon} > c > 0$.

Lemma 4 will be required in the proof of

LEMMA 5. *Let τ_Q be the first time that $\rho_t \geqslant Q$. Then, for h sufficiently large, there is a $q \in (0, 1)$ and a constant C_1 such that*

$$P\{\rho_{t+\tau_Q} > C_1(t+h)^{2/4-\epsilon} \text{ for } t > 0 \mid X_{\sigma(\tau_Q)}\} > q$$

uniformly in $X_{\sigma(\tau_Q)}$.

LEMMA 6. *Let $r = \inf(r(x), r(y))$ for $x, y \in N$. If $|\Theta(x, y)|$ denotes the angular distance between $\theta(x)$ and $\theta(y)$, then for some constants C and R, $r > R$,*

$$|\Theta(x, y)| \leqslant C\frac{{}^N d(x, y)}{\exp(r^\epsilon)},$$

where ${}^N d$ denotes the distance in the manifold N.

PROOF. We use the Rauch comparison theorem for the same comparison manifold P as is used in the proof of Lemma 2.

LEMMA 7. *There exists a random time T such that for all $m > T$,*

$$\sup_{\sigma(m)\leqslant t\leqslant\sigma(m+1)} {}^N d(F(X_t), F(X_{\sigma(m)})) \leqslant m.$$

PROOF. Using Lemma 3, we must show, for $m > T$, that $\sup_{m\leqslant t\leqslant m+1}\rho_t^{(m)} \leqslant m$. The argument is similar to an idea of Prat [11], and is also used by Kendall [7] and Pinsky [10].

LEMMA 8. *Fix $\delta > 0$. If τ is a stopping time, we may choose an integer J so large that*

$$P\{\sup_{\tau \leqslant t-m \leqslant \tau+1} \rho_t^{(\tau+m)} \leqslant m + J, \text{ for all } m \geqslant 1\} \geqslant 1 - \delta.$$

Lemmas 4–8, and a result of Stroock and Varadhan [**12**, Theorem 3.1] show that the tail σ-field $\lim_{t\to\infty} \theta(F(X_t))$ of X_t exists and is nontrivial. This is a contradiction, so F must be constant.

3. **Picard's theorem for Riemann surfaces of large genus.** In the sequel, let M and N be compact Riemann surfaces with a finite number of points deleted. Any such surface N is homeomorphic to a sphere with $t(N)$ tori attached and $p(N)$ points deleted. Let $n(N) = 2t(N) + p(N)$.

THEOREM 2. *Let F: $M \longrightarrow N$ be a holomorphic map. If $t(M) > 0$, $p(M) > 0$, $p(N) > 1$, and $n(N) - n(M) > 0$, then F is constant.*

PROOF OF THEOREM 2. The proof uses Brownian motion on Riemann surfaces, which can be defined as follows. Let $\{m_i\}$ be coordinate patches on M, and for each i, let c_i: $m_i \longrightarrow \mathbf{C}$, where $\mathbf{C}$ is the complex plane, be a holomorphic map. For $p \in M$, choose a patch $m_{i(1)}$ containing p. Let $W(t)$ be Brownian motion in $\mathbf{C}$, and let σ_1 be the first time that $p + W(t)$ hits the boundary of $c_{i(1)}(m_{i(1)})$. For $0 \leqslant t \leqslant \sigma_1$, let $B(t) = c_{i(1)}^{-1}(p + W(t))$. Next, let $m_{i(2)}$ be a patch containing $B(\sigma_1)$, and let σ_2 be the first time that $c_{i(2)}(B(\sigma_1)) + W(t) - W(\sigma_1)$ hits the boundary of $m_{i(2)}$. For $\sigma_1 \leqslant t \leqslant \sigma_2$, let

$$B(t) = c_{i(2)}^{-1}(c_{i(2)}(B(\sigma_1)) + W(t) - W(\sigma_1)),$$

and note that we have defined $B(t)$ to be continuous at σ_1. For $k > 2$, define σ_k and $B(t)$, $\sigma_k \leqslant t \leqslant \sigma_{k+1}$, analogously.

We also need

LEMMA 9. *Let $p(N) > 0$. Then the fundamental group of N is the free group with $n(N) - 1$ generators. The generators may be taken to be the cycles around single points (except one) and the canonical generators of the tori.*

Suppose F is not constant. Let $G(M)$ be the fundamental group of M with base point p, and let $G(N)$ be the fundamental group of N with base point $F(p)$. Let $\{\alpha_i^M\}$, $\{\alpha_i^N\}$ be the generators of $G(M)$, $G(N)$, resp., as described in Lemma 9. Choose 2 generators α_1^M, α_2^M of $G(M)$ which are generators of a torus, and let $\alpha_3^M, \ldots, \alpha_{n(M)-1}^M$ be the remaining generators. Let $H(M)$ be the smallest normal subgroup of $G(M)$ containing $\alpha_3^M, \ldots, \alpha_{n(M)-1}^M$ and the commutator $[G(M), G(M)]$. Let $\hat{G}(M) = G(M)/H(M)$, and note that $\hat{G}(M)$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$.

Let $\mathcal{O}$, $\mathcal{O}_2$, $\mathcal{O}_3$ be neighborhoods of p, contained in $m_{i(1)}$, whose images under $c_{i(1)}$ are discs of radii ϵ, 2ϵ, 3ϵ, respectively. Let $B^Q(t)$ be the Brownian motion on M whose initial distribution $B(0)$ is the measure Q, concentrated on the boundary of $\mathcal{O}$. Let $\tau_0 = 0$; given τ_i, let τ_{i+1} be the first time after τ_i for which $B(\tau_{i+1}) \in \partial\mathcal{O}$, and $\{B(t)\colon \tau_i \leqslant t \leqslant \tau_{i+1}\}$ corresponds to a nontrivial element of $\hat{G}(M)$. Note that since M is compact except for deleted points, all τ_i are finite.

Next, let $x_n = B^Q(\tau_n)$, and note that x_n is a Markov process with respect to the fields $\sigma\{B(t)\colon t \leqslant \tau_n\}$. Let m be the measure on $\partial\mathcal{O}$ induced by Lebesgue measure on the circle $c_{i(1)}(\partial\mathcal{O})$. By a theorem of Harris [6], it can be shown that x_n has an invariant measure. We set Q equal to this measure.

Let $\hat{X}_i$ be the element of $\hat{G}(M)$ corresponding to $\{B^Q(t)\colon 0 \leqslant t \leqslant \tau_i\}$. Since $\hat{G}(M) \cong \mathbf{Z} \times \mathbf{Z}$, we may regard $\hat{X}_i$ as a random walk on $\mathbf{R}^2$. Let $\Delta_i = \hat{X}_i - \hat{X}_{i-1}$. Then, $\{\Delta_i\}$ is a stationary process with $E|\Delta_i|^4 < \infty$. Moreover the process is symmetric, so Δ_i and $-\Delta_i$ have the same distribution. Using the central limit theorem (see Theorem 9 of Phillip [9]), it is shown that $\hat{X}_i$ is recurrent. By Lemma 3.1 of [1], it follows that X_i is recurrent.

Now, F induces a map F_H from $H/[H, H]$ to $G(N)/[G(N), G(N)]$. Since these are commutative groups with difference in dimension at least 3, it follows that there must be at least 3 generators α_1^N, α_2^N, α_3^N of $G(N)$ which generate a subgroup of $G(N)/[G(N), G(N)]$ modulo $F_H(H/[H, H])$ isomorphic to $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$. Let $H(N)$ be the smallest normal subgroup of $G(N)$ containing $\alpha_4^N, \ldots, \alpha_{n(N)-1}^N$ and $[G(N), G(N)]$, and let $\hat{G}(N) = G(N)/H(N)$. Then, F induces a map $\hat{F}$ from $\hat{G}(M)$ to $\hat{G}(N)$.

Using Brownian motion on N, we construct as before an invariant measure Q on $F(\mathcal{O})$ and a random walk $\hat{Y}_i$ from $F(B(t))$. By a theorem of Lévy [1], $F(B(t))$ is Brownian motion on N with a new time scale. By the Borel-Cantelli lemma, $\hat{Y}_i$ is transient. Davis's argument [1] then shows that the random walk Y_i induced by $F(B(t))$ with $B(0) = p$ is also transient. But then $\hat{F}(X_i) = Y_i$, and X_i is recurrent. This contradiction shows that F must be constant.

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