

## THE RADICAL IN A FINITELY GENERATED P.I. ALGEBRA

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Let  $R$  be an associative ring over a commutative ring  $\Lambda$ ,  $p\{X_1, \dots, X_e\}$  a polynomial on the free noncommuting variables  $X_1, \dots, X_e$ , with coefficients in  $\Lambda$  where one of its coefficient is  $+1$ . We say that  $R$  is a P.I. (polynomial identity) ring satisfying  $p\{X_1, \dots, X_e\}$  if  $p(r_1, \dots, r_e) = 0$  for all  $r_1, \dots, r_e$  in  $R$ .

We have the following

**THEOREM A.** *Let  $R = \Lambda\{x_1, \dots, x_k\}$  be a p.i. ring, where  $\Lambda$  is a noetherian subring of the center  $Z(R)$  of  $R$ . Then,  $N(R)$ , the nil radical of  $R$ , is nilpotent.*

Recall that  $N(R) = \bigcap_p P$  where the intersection runs on all prime ideals of  $R$ .

We obtain, as a corollary, by taking  $\Lambda$  to be a field, the following theorem, answering affirmatively the open problem which is posed in [Pr, p. 186].

**THEOREM B.** *Let  $R$  be a finitely generated P.I. algebra over a field  $F$ . Then,  $J(R)$ , the Jacobson radical of  $R$ , is nilpotent.*

This result, in turn, has the following important consequence.

**THEOREM C.** *Let  $R = F\{x_1, \dots, x_k\}$  be a finitely generated P.I. algebra over the field  $F$ . Then,  $R$  is a subquotient of some  $n \times n$  matrix ring  $M_n(K)$  where  $K$  is a commutative  $F$ -algebra. Equivalently, there exists an  $n$  such that  $R$  is a homomorphic image of  $G(n, t)$  the ring of  $t, n \times n$  generic matrices.*

Kemer, in [K], announced a proof of Theorem B with the additional assumption that  $\text{char}(F) = 0$ . His proof relies on a result of Razmyslov [Ra, Theorem 3] and on certain arguments related to the connection between P.I. ring theory and the theory of representation of the symmetric group  $S_n$  over  $F$ ,  $\text{char } F = 0$ . Both results rely heavily on the assumption that  $\text{char } F = 0$ , so they do not seem to generalize directly to arbitrary  $F$ .

The previously best known results concerning Theorem A are in [Ra, Theorems 1, 3, Sc, Theorem 2].

The proof of Theorem C is a straightforward application of Theorem B and a theorem of J. Lewin [Le, Theorem 10].

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We sketch a proof of Theorem B.

A major tool in our proof is the following result of Latyshev [La, Proposition 12]: “Let  $R$  be a p.i.  $F$ -algebra and  $I \subset N(R)$  a finitely generated two sided ideal. Then,  $I$  is nilpotent.” Latyshev’s result is originally stated with the additional assumption  $\text{char } F = 0$ , but it is superfluous.

We sketch the proof of Theorem B; the complete proof will appear elsewhere.

We have  $N(R) = P_1 \cap \cdots \cap P_t$ , where  $P_i$  are the minimal prime ideals of  $R$ , ordered via

$$\text{p.i.d.}(R/P_1) = \cdots = \text{p.i.d.}(R/P_m) \underset{\neq}{>} \text{p.i.d.}(R/P_{m+1}) \geq \cdots \geq \text{p.i.d.}(R/P_t)$$

where  $m \leq t$  (if  $m = t$ ,  $P_{m+1} \equiv R$ ). Here  $\text{p.i.d.}(S)$  denotes the minimal size of matrices into which  $S$  can be embedded.

Let  $\pi(R) = \text{p.i.d.}(R/P_1)$ ,  $d(R) = \max\{\text{k.d.}(R/P_i) \mid i = 1, \dots, m\}$  where  $\text{k.d.}(S)$  is the classical Krull dimension of  $S$ . One observes that there exists a  $b = b(k, d)$  such that if  $S$  is an  $F$ -algebra satisfying  $p(X_1, \dots, X_e)$  (of degree  $d$ ) and  $S = F\{y_1, \dots, y_k\}$  then  $\text{k.d.}(S) \leq b < \infty$ . We argue on the ordered pair  $\langle \pi(R), d(R) \rangle$  ordered lexicographically, that  $R$  is a counterexample to the theorem with minimal  $\langle \pi(R), d(R) \rangle$ . This will imply that there exists a  $\lambda \notin P_1 \cup \cdots \cup P_m$ , a finite sum of evaluations of some central polynomial of  $\pi \times \pi$  matrices ( $\pi \equiv \pi(R)$ ).

Using the result of Latyshev quoted above we may assume that  $\lambda \in Z(R)$  and by the minimal choice of  $R$ , since  $\langle \pi(R/\lambda R), d(R/\lambda R) \rangle < \langle \pi(R), d(R) \rangle$ , we get that  $\lambda^l R \subseteq N(R)$  for some  $l$ . Using Latyshev’s result once more we may assume that  $R_\lambda$ , the localization of  $R$  with respect to the set  $\{\lambda, \lambda^2, \dots\}$ , is Azumaya of rank  $\pi^2$  over its center. This in turn implies that  $\lambda^e R \subseteq Zb_1 + \cdots + Zb_h \equiv A$ , where  $Z \equiv Z(R)$ ,  $b_i \in R$ ,  $i = 1, \dots, h$  and  $A$  is a ring, for some  $e$ .

Consequently, we may assume that  $R$  satisfies any preassigned finite set of identities of  $\pi \times \pi$  matrices. Finally, an argument mimicing the argument appearing in [Ra, Theorem 3] enables us to settle this case.

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