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THE MAY-WIGNER STABILITY THEOREM FOR CONNECTED MATRICES

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In 1972 R. M. May [4], following empirical results of M. R. Gardner and W. R. Ashby [2], sketched a proof of the following asymptotic stability theorem for large random linear differential systems:

$$dx/dt = Ax. \tag{1}$$

May used deep results of E. P. Wigner [7], the "semicircle law" for eigenvalues of random symmetric matrices; see also Mehta [6].

MAY-WIGNER STABILITY THEOREM. *Let B be an $n \times n$ matrix with n^2C $(0 < C < 1)$ randomly located nonzero entries, each chosen independently from a symmetric distribution with variance α^2. Let $A = B - I$, and let $P(\alpha, n, C)$ be the probability that the corresponding differential system* (1) *has a stable equilibrium at* 0. *Let $\epsilon > 0$. Then $P(\alpha, n, C) \longrightarrow 1$ as $n \longrightarrow \infty$ provided $\alpha^2 nC < 1 - \epsilon$; conversely, $P(\alpha, n, C) \longrightarrow 0$ as $n \longrightarrow \infty$ for $\alpha^2 nC > 1 + \epsilon$.*

This result provided a basis for studying the stability of neutral models in both cybernetics [2] and ecology (May [5] and references therein). Unfortunately Wigner's results are extremely complex, and may not apply to all random matrices (G. Sugihara, private communication, see also [6, p. 150]). We therefore sought a conceptually simpler proof.

We announce here a direct proof for matrices with connected underlying graphs. More precisely, the underlying graph of A (with one edge joining i and j if A_{ij} or A_{ji} is nonzero) is asymptotically almost surely connected if

$$C \geqslant (1 + \epsilon)\log n/n,$$

and asymptotically almost surely not connected if

$$C \leqslant (1 - \epsilon)\log n/n$$

for any fixed positive ϵ, (Bollobás [1, p. 143]). We assume the former condition holds; in particular the theorem holds for any constant C.

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There are three main steps in the proof. First, the columns of A approach orthogonality rapidly as $n \longrightarrow \infty$. Secondly, for $\alpha^2 nC < 1 - \epsilon$, ϵ fixed and positive, each column has l^2 norm asymptotically almost surely less than $1 - 3\epsilon^2/4$; a suitable converse bound also holds $\alpha^2 nC > 1 + \epsilon$. Finally, the Gerschgorin bound on the largest eigenvalue of a matrix is readily extended to a l^2 analogue for matrices with orthogonal columns.

A detailed proof and ecological applications and extensions will appear elsewhere [3]. We thank Drs. R. M. May and G. Sugihara for helpful conversations.

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