THE MAY-WIGNER STABILITY THEOREM
FOR CONNECTED MATRICES
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In 1972 R. M. May [4], following empirical results of M. R. Gardner and W. R. Ashby [2], sketched a proof of the following asymptotic stability theorem for large random linear differential systems:

\[ \frac{dx}{dt} = Ax. \]

May used deep results of E. P. Wigner [7], the "semicircle law" for eigenvalues of random symmetric matrices; see also Mehta [6].

**May-Wigner Stability Theorem.** Let \( B \) be an \( n \times n \) matrix with \( n^2 C (0 < C < 1) \) randomly located nonzero entries, each chosen independently from a symmetric distribution with variance \( \sigma^2 \). Let \( A = B - I \), and let \( P(\alpha, n, C) \) be the probability that the corresponding differential system (1) has a stable equilibrium at 0. Let \( \epsilon > 0 \). Then \( P(\alpha, n, C) \to 1 \) as \( n \to \infty \) provided \( \alpha^2 n C < 1 - \epsilon \); conversely, \( P(\alpha, n, C) \to 0 \) as \( n \to \infty \) for \( \alpha^2 n C > 1 + \epsilon \).

This result provided a basis for studying the stability of neutral models in both cybernetics [2] and ecology (May [5] and references therein). Unfortunately Wigner's results are extremely complex, and may not apply to all random matrices (G. Sugihara, private communication, see also [6, p. 150]). We therefore sought a conceptually simpler proof.

We announce here a direct proof for matrices with connected underlying graphs. More precisely, the underlying graph of \( A \) (with one edge joining \( i \) and \( j \) if \( A_{ij} \) or \( A_{ji} \) is nonzero) is asymptotically almost surely connected if

\[ C > (1 + \epsilon) \log n/n, \]

and asymptotically almost surely not connected if

\[ C < (1 - \epsilon) \log n/n \]

for any fixed positive \( \epsilon \), (Bollobás [1, p. 143]). We assume the former condition holds; in particular the theorem holds for any constant \( C \).
There are three main steps in the proof. First, the columns of $A$ approach orthogonality rapidly as $n \to \infty$. Secondly, for $\alpha^2 nC < 1 - \varepsilon$, $\varepsilon$ fixed and positive, each column has $l^2$ norm asymptotically almost surely less than $1 - 3\varepsilon^2/4$; a suitable converse bound also holds $\alpha^2 nC > 1 + \varepsilon$. Finally, the Gerschgorin bound on the largest eigenvalue of a matrix is readily extended to a $l^2$ analogue for matrices with orthogonal columns.

A detailed proof and ecological applications and extensions will appear elsewhere [3]. We thank Drs. R. M. May and G. Sugihara for helpful conversations.

REFERENCES


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