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EXOTIC CLASSES FOR MEASURED FOLIATIONS

BY STEVEN HURDER

A measured foliation (F, μ) is a C^2-foliation F on a smooth manifold M and a transverse invariant measure μ for F [14]. Inspired by the foliation index theorem of Connes [4, 5], we study the result of integrating *normal data* to F over the leaf space M/F. This produces new secondary-type exotic classes for measured foliations [7]. These classes have applications to SL_q-foliations, to the study of groups of volume-preserving diffeomorphisms, and also are useful for relating the geometry of F to the values of the usual secondary classes [8, 9].

THEOREM 1. *Let (F, μ) be a measured foliation of codimension q on M. If either M is closed and orientable, or μ is absolutely continuous (so it is represented by a closed form $d\mu$), then there is a well-defined characteristic map*

$$\chi_\mu: H^*(\mathrm{gl}_q, O_q) \longrightarrow H^{*+q}(M).$$

We call the image of χ_μ the μ-classes of (F, μ).

For M^m compact and $y_I \in H^n(\mathrm{gl}_q, O_q)$, the class $\chi_\mu(y_I)$ is defined as the geometric current in $H_{m-n-q}(M)$ obtained by integrating over the leaf space of F, via μ, the leaf classes corresponding to y_I. Duality then produces the invariant in $H^{n+q}(M)$. If $d\mu$ is a closed form representing μ, then a cocycle representing $\chi_\mu(y_I)$ is $\Delta(y_I) \cdot d\mu$, where Δ: $WO_q \longrightarrow A^{\cdot}(M)$ is the secondary map for F, [2, 10]. Complete details and properties of χ_μ are described in [7].

The values of the μ-classes depend on the measure μ and the dynamical behavior of F in a neighborhood of the support of μ. It is conjectured that subexponential growth of the leaves of F implies the μ-classes vanish; this can be shown in some cases. Examples can be constructed for which all of the μ-classes are nontrivial.

The canonical measure associated to an SL_q-foliation (F, ω)—where ω is a transverse invariant volume form—defines a characteristic map χ_ω: $H^*(\mathrm{sl}_q, SO_q) \longrightarrow H^{*+q}(M)$, and these come from universal classes for the Haefliger classifying space $B\Gamma_{SL_q}$. There are additional μ-classes for measured foliations with framed normal bundles, and corresponding universal classes for $B\overline{\Gamma}_{SL_q}$, the homotopy fiber of $B\Gamma_{SL_q} \longrightarrow BSL_q$.

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THEOREM 2. *The characteristic maps*

$$\chi\colon H^n(\mathrm{sl}_q, SO_q) \longrightarrow H^{n+q}(B\Gamma_{SL_q}),$$

$$\chi^s\colon H^n(\mathrm{sl}_q) \longrightarrow H^{n+q}(B\overline{\Gamma}_{SL_q})$$

are injective for $n < [(q-1)/4]$. *Further,* χ^s *is injective on the image* $H^*(\mathrm{so}_q) \longrightarrow H^*(\mathrm{sl}_q)$.

To show nontriviality, we compute the values of the μ-classes for the foliations obtained by suspending the action of SL_qZ on T^q. This type of example was suggested to us by W. Thurston. The corresponding characteristic map is related to the Van Est map of continuous cohomology $H^*(\mathrm{sl}_q, SO_q) \longrightarrow H^*(SL_qZ; \mathbf{R})$, which is injective in degrees less than $[(q-1)/4]$ by Borel **[1]**. Note that χ^s is nontrivial for all $q \geqslant 3$, but Theorem 2 only asserts χ is nontrivial for $q \geqslant 25$. We do not know of SL_q-foliations with small codimension and nontrivial μ-classes from χ.

Let ω be a volume form on $\mathbf{R}^q$ with infinite total mass and $\mathrm{Diff}^c_\omega\mathbf{R}^q$ the group of compactly supported diffeomorphisms which preserve ω. We use McDuff's generalization of the Mather-Thurston theorem **[11]** and Theorem 2 to prove

COROLLARY 1.. *There are inclusions of* $\mathbf{Q}$*-vector spaces*

$$H_n(\mathrm{so}_q; \mathbf{R}) \hookrightarrow H_n(B\overline{\mathrm{Diff}^c_\omega}\mathbf{R}^q; \mathbf{Q})$$

for $n < q$, *and* $\mathbf{R} \subseteq H_3(B\overline{\mathrm{Diff}^c_\omega}\mathbf{R}^q; \mathbf{Q})$ *for all* $q \geqslant 3$.

It was shown that $H_1(B\overline{\mathrm{Diff}^c_\omega}\mathbf{R}^q; Z) = 0$ for $q > 2$ by Thurston-Banyaga. Corollary 1 gives the first nonvanishing results for the group homology in degrees less than $q + 1$; in degrees $\geqslant q + 1$, the secondary classes of SL_q-foliations detect nontrivial homology of $B\overline{\mathrm{Diff}^c_\omega}\mathbf{R}^q$. McDuff has investigated in **[12]** the geometrical significance of some of these new invariants for $\mathrm{Diff}^c_\omega\mathbf{R}^q$ and also defined further interesting classes.

The residuable secondary classes are the cocycles y_Ic_J in $H^*(WO_q)$ with degree $c_J = 2q$ maximal. The "integration over the fiber" process is faithful on these classes, so a residue theory can be developed for them. Given a measured foliation $(\mathcal{F}, \mu)$ with support $\mu = M$, the residuable classes decompose into the measure class $d\mu$ product with a leaf invariant. This observation can be used to relate the residuable secondary classes with the geometry of $\mathcal{F}$.

THEOREM 3. *Let* $\mathcal{F}$ *be a codimension* q *compact foliation* (*that is, each leaf of* $\mathcal{F}$ *is compact*) *on a closed manifold* M. *Each residuable secondary class* $\Delta_*(y_Ic_J) \in H^*(M)$ *is then zero.*

The idea of the proof is to integrate $\Delta(y_I c_J)$ over M, decompose this integral over saturated sets—the *Epstein filtration* of the bad set—where each saturated set has a transverse invariant measure of maximal support. Each integral decomposes into a weighted sum of leaf classes, and then we show the leaf classes for a compact foliation uniformly vanish. Details appear in [8].

For codimension one foliations remarkable progress has been made in relating the geometry of a foliation with its Godbillon-Vey invariant [3, 13]. For higher codimensions, it is expected that a geometric interpretation of the residuable secondary classes can be achieved by utilizing the techniques of the proof of Theorem 3, the residue theorem for foliations [6] and the properties of the μ-classes. Some progress on this problem is given in [9].

ADDED IN PROOF. G. Duminy has recently proved that a codimension-one foliation on a compact manifold with nonvanishing Godbillon-Vey invariant must have a resilient leaf (*L'Invariant de Godbillon-Vey d'un feuilletage se localise dans les feuilles ressort,* preprint.)

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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08544