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Stochastic filtering theory, by G. Kallianpur, Springer-Verlag, New York, Heidelberg, Berlin, 1980, xvi + 316 pp.,$29.80.

An important problem in statistical communication theory is the separation of random signals from random noise. These phenomena are modelled by stochastic processes s_t, n_t called respectively the signal and noise processes. The signal cannot be observed directly; instead, at time t the sum

$$z_t = s_t + n_t \tag{1}$$

is observed. Roughly speaking, the filtering problem is to make a "best" estimate for s_t given observations z_τ for times $\tau \leq t$. Closely related problems are to best estimate s_T when $t < T$ (the prediction problem) and for $T < t$ (the data smoothing problem). By "best" estimate $\hat{s}_t$ let us mean an estimate minimizing the mean squared error $E(s_t - \hat{s}_t)^2$, with $E(-)$ denoting expected value. Pioneering work on the filtering problem was done by Wiener and Kolmogorov during the 1940s. In that work, the filtering problem was considered in the frequency domain, by taking Fourier transforms of z_t, s_t, n_t. The problem was reduced to solving an integral equation of Wiener-Hopf type.

Linear filtering theory took a distinctive new direction around 1960, stimulated by two key papers by Kalman [**9**] and Kalman and Bucy [**10**]. In their approach the filtering problem is considered in the time domain (rather than the frequency domain), and state space representations are introduced. The signal is expressed as a linear function of an N-dimensional state vector X_t, namely, $s_t = CX_t$. In the continuous-time Kalman-Bucy model, the state process X_t evolves according to the differential equations

$$\dot{X}_t = AX_t + \nu_t, \qquad t \geq 0, \tag{2}$$

with initial state X_0 some gaussian random vector and with ν_t a white noise process. Equation (1) takes the form

$$z_t = CX_t + n_t, \tag{3}$$

with the observation noise n_t also assumed white, and with X_0, ν, n independent. A white noise process ν_t is supposed to be a stationary, gaussian process whose values ν_s, ν_t at different times s, t are independent. One should also have $\nu_t = d\beta_t/dt$, where β_t is a brownian motion process. However, with probability 1 the brownian sample paths $\beta.$ are continuous but nowhere differentiable functions of t. In the linear filter model this difficulty can be avoided by interpreting equations (2), (3) in a generalized sense (e.g. in terms of Schwartz distributions). However, for the nonlinear filter problem with which Kallianpur's book is mainly concerned, stochastic differential equations provide a convenient framework for nonlinear filtering theory. The formal nonlinear equation (5) below corresponding to (2) is then rewritten as the stochastic differential equation (7).

For the linear Kalman-Bucy model, there is a rather simple and elegant solution to the filter problem: $\hat{s}_t = C\hat{X}_t$, where $\hat{X}_t$ obeys a linear differential equation similar to (2):

$$\dot{\hat{X}}_t = A\hat{X}_t + F_t\hat{\nu}_t, \tag{4}$$

with $\hat{\nu}_t = z_t - C\hat{X}_t$ another white noise (called the innovation.) The error $s_t - \hat{s}_t$ is gaussian and independent of the observations up to time t. Moreover, both the coefficient F_t in (4) and the error covariance matrix can be precomputed by solving a matrix Riccati equation. The technique has been widely applied. See for example [7].

The simplicity of the Kalman-Bucy filter results depends on the linearity of equations (2), (3) and on the gaussian nature of all stochastic processes involved. However, in practice problems are often encountered for which the linear-gaussian model is not a good approximation. The extensive literature on nonlinear filtering (theoretical and applied) which has appeared since 1965 is concerned with this question. Kallianpur's book gives a treatment of the basic nonlinear filter equation, together with the rather sophisticated probabilistic tools needed to derive it in a mathematically rigorous way. A more comprehensive development of this material appears in [**11**]. However, Kallianpur's more concise treatment may be more accessible to nonspecialists.

After a brief introductory chapter, Kallianpur summarizes parts of the theory of continuous time martingales and semimartingales. Proofs of some more technical results are omitted, with references to [**12**] and occasionally other sources. Next come three chapters of rather standard material on stochastic integrals with respect to a square integrable martingale (the Itô integral with respect to brownian motion is a special case) and to stochastic differential equations. These are followed by two more tools needed later in deriving the nonlinear filter equation. One of these is Girsanov's theorem for change of probability measure corresponding to change of drift in a stochastic differential equation. The other concerns functionals of a brownian motion process, include the following beautiful result. Let W be a brownian motion, and $\mathcal{F}_T(W)$ the least σ-algebra with respect to which the random variables W_t are measurable for $0 \leqslant t \leqslant T$. Let $\xi \in L^2[\mathcal{F}_T(W)]$ with $E\xi = 0$. Then ξ is the sum of an L^2-convergent series $\xi = \xi_1 + \xi_2 + \cdots + \xi_p + \cdots$, of orthogonal random variables, with ξ_p a p-fold multiple Wiener integral corresponding to Wiener's p-fold homogeneous chaos. The reviewer found it illuminating to read K. Itô's original treatment of this result [8] along with Kallianpur's more general (and somewhat formalized) development.

In the nonlinear filtering problem, to avoid undue complication, let us consider a model in which (2), (3) are replaced by nonlinear equations

$$\dot{X}_t = b(X_t) + \sigma(X_t)\nu_t, \tag{5}$$

$$z_t = h(X_t) + n_t, \tag{6}$$

with bounded and sufficiently smooth functions b, σ, h. Actually, to formulate (5), (6) in a mathematically precise way, we introduce independent brownian motions β_t, B_t whose formal time-derivatives are the white noises ν_t, n_t. Then (5) is interpreted as the Itô-sense stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)d\beta_t, \tag{7}$$

and (6) becomes

$$dZ_t = h(X_t)dt + dB_t, \qquad Z_0 = 0, \tag{8}$$

with z_t the formal time-derivative of Z_t. The signal is now $\hat{s}_t = h(X_t)$. It is easily seen that the mean square optimal estimate $\hat{s}_t$ for s_t is the conditional expectation of $h(X_t)$ with respect to the σ-algebra $\mathcal{F}_t(Z)$. The problem is to describe the dynamics of $\hat{s}_t$. Unlike the linear-gaussian case, one generally needs to know the conditional distribution of X_t to find $\hat{s}_t$. This makes the nonlinear filter problem an infinite-dimensional one. Under suitable assumptions (including nonsingularity of the matrix σ in (5)) the conditional distribution has a density $p(x, t)$, and then

$$\hat{s}_t = \int_{R^n} xp(x, t)\, dx. \tag{9}$$

The density $p(x, t)$ satisfies a nonlinear stochastic partial differential/integral equation, called the nonlinear filter equation. The stochastic effects enter the nonlinear filter equation through the so-called innovation brownian motion

$$\hat{B}_t = Z_t - \int_0^t h(x)p(x, t)\, dx. \tag{10}$$

The proof given for the nonlinear filter equation follows generally the original derivation in the fundamental paper [6]. A crucial step is to show that any square integrable $\mathcal{F}_t(Z)$ martingale of mean 0 is a stochastic integral with respect to the innovation process $\hat{B}_t$. If one knows that the Kailath innovations conjecture $\mathcal{F}_t(Z) = \mathcal{F}_t(\hat{B})$ is true, then this follows directly from the multiple

Wiener integral expansion mentioned above. This conjecture was settled affirmatively only in 1980 [**1**]. However, this point can be avoided by making a Girsanov transformation after which the observation process Z_t is a brownian motion, and by using a stopping time argument.

There have been a number of interesting recent developments in nonlinear filtering theory, which are beyond the scope of Kallianpur's book. One direction concerns the theory of "robust" or "pathwise" solutions to the filtering equations [**4**]. The objective is to obtain $\hat{s}_t$ for all possible observation trajectories $Z.$, not just for a set of probability 1, in such a way that $\hat{s}_t$ depends continuously on $Z.$ in the uniform norm. Another direction of recent research is to explain the structure of the optimal filter by studying a certain Lie algebra associated with it [**3**]. A related problem is to find finite-dimensional nonlinear filters, in other words, filters whose evolution in time is described by a finite number of stochastic differential equations [**2**].

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Perturbation methods in applied mathematics, by J. Kevorkian and J. D. Cole, Applied Mathematical Sciences, vol. 34, Springer-Verlag, Berlin and New York, 1981, x + 558 pp., $42.00.

Introduction to perturbation techniques, by Ali Hasan Nayfeh, Wiley, New York, 1981, xiv + 519 pp., $29.95.

Singular perturbations, in 1982, is a maturing mathematical subject with a fairly long history and a strong promise for continued important applications throughout science. Though the basic intuitive ideas involving local patching of