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Circulant matrices, by Philip J. Davis, Wiley, New York, 1979, xv + 250 pp., $18.95.

As mathematics seemingly stretches out to infinity in all directions, the pedagogical urge grows within some of us to take one simple mathematical concept, to turn it over and examine it from many viewpoints, and to study it in relation to several areas of mathematics. It is this that Philip Davis has done, in an interesting and illuminating way, in *Circulant matrices*.

Circulants are $n \times n$ complex matrices of the form

$$C = \operatorname{circ}(c_1, c_2, \ldots, c_n) = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ & & \cdots & \\ c_2 & c_3 & \cdots & c_1 \end{pmatrix};$$

they first appeared, as examples, in a 1846 paper by E. Catalan. The main theorem on complex circulants seem to be that they can be unitarily diagonalized using the Fourier matrix

$$F = \frac{1}{\sqrt{n}}\left(z^{(j-1)(k-1)}\right),$$

where $z = \exp(-2\pi i/n)$. Thus, the set of $n \times n$ circulants form a commutative algebra. Also, the Moore-Penrose inverse of a circulant is a circulant. As the author notes in his Preface, "practically every matrix-theoretic question for circulants may be resolved in 'closed form'. Thus the circulants constitute a nontrivial but simple set of objects that the reader may use to practice, and ultimately deepen, a knowledge of matrix theory." In this book, many matrix concepts are introduced, and then discussed for circulants, both in the text and in the numerous exercises. Generalizations to g-circulants, block circulants, magic squares, and centralizers afford additional opportunities for such practice.

The principal application presented is to nested polygons. As an example, let Q denote both a column vector of complex numbers z_1, z_2, z_3, z_4 and the quadrilateral in the complex plane whose vertices are, in order, z_1, z_2, z_3, z_4. The parallelogram formed by joining the midpoints of adjacent sides of Q is $P = DQ$, where $D = \operatorname{circ}(.5, .5, 0, 0)$. The behavior of the sequence of nested parallelograms $P = DQ$, $DP = D^2Q, \ldots$ is analyzed by studying the behavior of the sequence of circulants $D, D^2, \ldots$. Also, given parallelogram P, there are infinitely many quadrilaterals Q for which $DQ = P$, but only one parallelogram $Q = D^{\div} P$, where $D^{\div}$ is the Moore-Penrose inverse of D.

Other applications within mathematics abound, to geometry (the isometric inequality and the discrete inequality of Wirtinger), and to analysis (infinite products of matrices, smoothing and variation reduction, and best approximation). Applications to group theory, and to areas outside mathematics, are mentioned in passing.

I enjoyed reading this book. While presenting a lot of mathematics, carefully and in considerable detail, Davis has managed to be conversational whenever possible, and to tell us where he's going and why. A class of advanced undergraduates with good linear algebra preparation should be able to read this book, to work the exercises, and to discuss their work together. Even though circulant matrices are themselves not of primary importance, their study, via this excellent book, could be a valuable part of discovering mathematics as a discipline.

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Essays in commutative harmonic analysis, by Colin C. Graham and O. Carruth McGehee, Grundlehren der Mathematischen Wissenschaften, Band 238, Springer-Verlag, New York, 1979, xxi + 464 pp., $42.00.

Harmonic Analysis has passeed through a series of distinct epochs. The subject began in 1753 when Daniel Bernoulli gave a "general solution" to the problem of the vibrating string, previously investigated by d'Alembert and Euler, in terms of trigonometric series. This first epoch continues up to Fourier's book (1822). It is not completely clear what was going on in the minds of the harmonic analysts of this period, but the dominant theme seems to me to be the spectral theory of the operator d^2/dx^2 on $\mathbf{T} = \mathbf{R}/\mathbf{Z}$.

In 1829 Modern Analysis emerges fully armed from the head of Dirichlet [**3**]. His proof that the Fourier series of a monotone function converges pointwise everywhere meets the modern standards of rigor. All needed concepts are defined, a marked contrast with previous habits, and the demonstration is expeditious. The operator d^2/dx^2 does not enter. The next epoch-making work is Riemann's *Habilitationschrift* [**6**]. Once more d^2/dx^2 plays the central role, albeit in a generalized form,

$$D^2_{\text{sym}} f(x) = \lim_{n\to\infty} h^{-2}[f(x+h) + f(x-h) - 2f(x)].$$

The big gap in Riemann's work is the Uniqueness Theorem, proved by Cantor [**1**] in 1870 after Schwarz has shown that the solution to $D^2_{\text{sym}} f = 0$ are harmonic functions in the ordinary sense. After this d^2/dx^2 goes its own way. With the Lebesgue integral and the pioneering work of Herman Weyl there has been a main branch of harmonic analysis going on for seventy years or so where the principal theme has been the study of various group-invariant generalizations of d^2/dx^2. Almost all of what is called "harmonic analysis on Lie groups" is in this vein. Moreover, this is the branch of the subject where exciting new ideas are still pouring forth. If one looks at what the leaders in the field have been doing recently, in the majority of cases one finds that harmonic