

NEW DEFECT RELATIONS
 FOR MEROMORPHIC FUNCTIONS ON C^n

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For meromorphic functions f and φ on the complex line C^1 , one considers the counting function $N(r, \varphi, f) = \int_1^r n(t, \varphi, f)t^{-1} dt$, where $n(t, \varphi, f)$ denotes the number of solutions of the equation $f = \varphi$ (counting multiplicities) on the disk $\{|z| \leq t\}$. If $T(r, \varphi) = o(T(r, f))$, one defines the defect $\delta(\varphi, f) = \liminf[1 - N(r, \varphi, f)/T(r, f)]$ and observes that $0 \leq \delta(\varphi, f) \leq 1$ as in the case where $\varphi = \text{constant}$. In 1929, R. Nevanlinna [4, p. 77] asked if the defect relation

$$(*) \quad \sum_{j=1}^q \delta(\varphi_j, f) \leq 2$$

is valid for distinct meromorphic functions φ_j with $T(r, \varphi_j) = o(T(r, f))$. The case where the φ_j are constant is Nevanlinna's fundamental defect relation [4]. (If $q = 3$, then $(*)$ follows immediately from the Nevanlinna defect relation.) In 1939, J. Dufresnoy [3] showed that $\sum \delta(\varphi_j, f) \leq d + 2$ if f is transcendental and the φ_j are distinct polynomials of degree $\leq d$. In 1964, C.-T. Chuang [2] gave a general Second Main Theorem which yields $(*)$ for the case where f is holomorphic (or more generally when $\delta(\infty, f) = 1$) and which generalizes the defect relation of Dufresnoy [3]. However, this question of Nevanlinna remains unanswered today even for polynomial φ_j , despite Nevanlinna's assertion [4, p. 77] that $(*)$ "follows easily" for this case. If f is a meromorphic function on C^n , then a special case of a theorem of W. Stoll [7] (see also Vitter [8]) yields $(*)$ for constant φ_j as in the classical Nevanlinna theory. (In fact, the results of Chuang [2] generalize easily to C^n .) In this note we announce a new defect relation of the form $(*)$ for meromorphic functions on C^n , $n \geq 2$.

If f and φ are distinct meromorphic functions on C^n , we let $D(\varphi, f)$ denote the divisor on C^n given by the solution (with multiplicities) to the equation $f = \varphi$. We write $N(r, \varphi, f) = N(r, D(\varphi, f))$, where $N(r, D)$ denotes the counting function for D as given in [1 or 7]. We easily obtain the *First Main Theorem*,

$$N(r, \varphi, f) + m(r, \varphi, f) = T(r, f) + T(r, \varphi) + c,$$

where the proximity term $m(r, \varphi, f) \geq 0$. Our main result is the following

SECOND MAIN THEOREM. *Let $f, \varphi_1, \dots, \varphi_q$ be distinct meromorphic functions on C^n ($q \geq n - 1$) such that*

- (i) $\text{rank}(\varphi_1, \dots, \varphi_q) = n - 1$,

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(ii) $\text{rank}(f, \varphi_1, \dots, \varphi_{n-1}) = n$.

Then

$$\| (q-2)T(r, f) \leq \sum_{j=1}^q N(r, \varphi_j, f) - N(r, R(f, \varphi_1, \dots, \varphi_{n-1})) + O\left(\sum_{j=1}^q T(r, \varphi_j) + \log r T(r, f)\right).$$

Here “rank” means the maximal rank of the derivative matrix, and R stands for the ramification divisor (given by the Jacobian determinant). The symbol $\|$ means that the inequality is valid for all $r > 0$ outside a set of finite Lebesgue measure. (If f is of finite order and the φ_j are rational, then the inequality of the theorem is valid for all $r > 0$.)

As in the classical theory, we let $\bar{N}(r, \varphi, f)$ denote the counting function obtained by reducing all multiplicities to 1. The Second Main Theorem can be restated as follows:

COROLLARY 1. *Let $f, \varphi_1, \dots, \varphi_q$ be distinct meromorphic functions such that*

$$\text{rank}(f, \varphi_1, \dots, \varphi_q) = \text{rank}(\varphi_1, \dots, \varphi_q) + 1.$$

Then

$$\| (q-2)T(r, f) \leq \sum_{j=1}^q \bar{N}(r, \varphi_j, f) + O\left(\sum_{j=1}^q T(r, \varphi_j) + \log r T(r, f)\right).$$

For example, if $\varphi_j = \varphi_j(z_1, \dots, z_{n-1})$ and $\partial f / \partial z_n \not\equiv 0$, then $f, \varphi_1, \dots, \varphi_q$ satisfy the hypothesis of Corollary 1. More generally, we can let $\varphi_j = \varphi_j(g_1, \dots, g_p)$ where $p \leq n-1$ and the g_k are meromorphic functions on \mathbb{C}^n such that $\text{rank}(f, g_1, \dots, g_p) = p + 1$.

If $T(r, \varphi) = o(T(r, f))$, then we define the defect $\delta(\varphi, f)$ as in the one variable case above, and we similarly let $\Theta(\varphi, f) = \liminf[1 - \bar{N}(r, \varphi, f)/T(r, f)]$. As in the classical case, we have $0 \leq \delta(\varphi, f) \leq \Theta(\varphi, f) \leq 1$. We now state our general defect relation, which follows immediately from Corollary 1.

COROLLARY 2 (DEFECT RELATION). *Let $f, \varphi_1, \dots, \varphi_q$ be as in Corollary 1. If $T(r, \varphi_j) = o(T(r, f))$ for $1 \leq j \leq q$, then*

$$\sum \delta(\varphi_j, f) \leq \sum \Theta(\varphi_j, f) \leq 2.$$

The proof of our Second Main Theorem uses the methods of [1 and 5] and the essential estimate given in the lemma below. We let

$$\omega = (\sqrt{-1}/2\pi)\partial\bar{\partial}\log(|w^0|^2 + |w^1|^2)$$

denote the Fubini-Study 2-form on $\mathbb{C}P^1$, and we let

$$\rho(a, b) = \frac{|a^1 b^0 - a^0 b^1|}{(|a^0|^2 + |a^1|^2)^{1/2} (|b^0|^2 + |b^1|^2)^{1/2}}$$

denote the chordal distance on CP^1 . We consider the function

$$\gamma = \rho^{-2}(4q - 2 \log \rho)^{-2} \text{ on } CP^1 \times CP^1$$

(γ blows up along the diagonal). Let $f, \varphi_1, \dots, \varphi_q$ be as in the Second Main Theorem, and assume $q \geq n + 2$. We regard f, φ_j as meromorphic maps into CP^1 . Let

$$S = \text{supp} \left[\sum_{j=1}^q D(\varphi_j, f) + R(f, \varphi_1, \dots, \varphi_{n-1}) \right].$$

We define the volume form Ψ on $C^n - S$ (which is a variant of the volume form given by Carlson and Griffiths [1]) by

$$\Psi = \left[\prod_{j=1}^q \gamma(\varphi_j, f) \right] f^* \omega \wedge \varphi_1^* \omega \wedge \dots \wedge \varphi_{n-1}^* \omega.$$

Recall that the Ricci form $\text{Ric } \Psi$ is given by $\text{Ric } \Psi = (\sqrt{-1}/2\pi) \partial \bar{\partial} \log h$ where $\Psi = h(\sqrt{-1} dz_1 \wedge d\bar{z}_1) \wedge \dots \wedge (\sqrt{-1} dz_n \wedge d\bar{z}_n)$. We let

$$\theta = \text{Ric } \Psi + 2 \sum_{i=1}^{n-1} \varphi_i^* \omega \text{ on } C^n - S.$$

LEMMA. θ is positive and $\theta^n > \lambda^{2q-2} \Psi$ on $C^n - S$, where

$$\lambda = \min_{j \neq k} \rho(\varphi_j, \varphi_k).$$

The positivity of θ is easy to verify without any condition on the rank of the φ_j . However the volume-form inequality of the lemma is not true in general if $\text{rank}(\varphi_1, \dots, \varphi_q) = n$. Details will appear in [6].

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