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Introduction to algebraic K-theory, by John R. Silvester, Chapman & Hall, London, 1981, xi + 255 pp., \$29.95 hardcover, \$13.95 paperback.

K -theory, both algebraic and topological, had its beginnings in the 1940s when Grothendieck defined a group, now known as the Grothendieck group, to use in his proof of the Riemann-Roch theorem. Algebraically, the Grothendieck group of a ring R is the abelian group generated by the projective R modules P with the relation $[P] = [P'] + [P'']$ whenever $0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0$ is exact. For example, if R is a field, a local ring or a principal ideal domain, then the Grothendieck group of R is \mathbf{Z} since the class of P depends only on its rank. Grothendieck's original definition had been for vector bundles over a space X and algebraists had generalized it by replacing vector bundles over X with projective modules over R . Thus K -theory seemed to divide into two camps, topological K -theory studying vector bundles and algebraic K -theory which studies rings. In both cases there is a set of functors mapping to abelian groups. The two theories are not very different and are often indistinguishable except for notation. Algebraic K -theorists often use topology in their work and topological K -theorists often use algebra. In fact, the most useful definition of higher algebraic K -groups is topological.

Although algebraic K -theory started as a set of functors from the category of rings to the category of abelian groups, it was soon generalized to a set of functors from an abelian category to abelian groups. Recently it has been generalized even further.

The Grothendieck group of R is $K_0(R)$. For each $i \geq 1$, the negative K -group, $K_{-i}(R)$, is a certain direct summand of K_0 of Laurent polynomials in i variables over R . Bass defined $K_1(R)$ to be $GL(R)/E(R)$, where $GL(R)$ is the group of all invertible matrices with entries in R and $E(R)$, the elementary group, is the subgroup of $GL(R)$ generated by $E_{ij}(r)$, $i \neq j$, $r \in R$, where $E_{ij}(r) = (a_{kl})$ with $a_{kk} = 1$, $a_{ij} = r$ and $a_{kl} = 0$ otherwise. Algebraically, K_1 studies how far the general linear group differs from the product of diagonal matrices and elementary matrices (they are equal for fields, division rings and local rings). In many ways K_1 is an attempt to generalize parts of linear algebra, especially the notions of dimension and determinant, to projective modules over an arbitrary ring. Milnor defined $K_2(R)$ which studies the relations in $E(R)$. Thus, K_1 and K_2 together determine the relations in the general linear group.

Some ring homomorphisms or sets of homomorphisms generate long exact sequences in K -theory, including Mayer-Vietoris sequences, localization sequences and the sequence of an ideal. These sequences were known for K_i , $i \leq 2$. Furthermore, in topological K -theory, there were definitions of $K_i^{\text{top}}(X)$, $i \geq 3$, that extended these sequences. Therefore, in the late sixties many people tried to define higher algebraic K -groups that would extend these sequences and agree with higher topological K -groups when appropriate. In the early

seventies Quillen solved the problem by giving two equivalent constructions for higher algebraic K -groups [4], both topological in nature, for which, among other results, he was honored with the Cole prize in 1975 and the Fields medal in 1978. Quillen's definitions of the higher K -groups are now the accepted ones because he proved that his definitions extend the sequences, agree with the topological definitions when appropriate, have transfer maps and many other desirable properties. Moreover, most of the other definitions, although not as powerful for proving the above theorems, are equivalent to his and thus are sometimes useful.

The algebraic objects, $K_i(R)$, $i \leq 2$, have applications in topology. For example, $K_2(\mathbf{Z}\pi)$ is used to measure obstructions to pseudo-isotopies being isotopies on a topological space with fundamental group π . Also, $K_{-i}(\mathbf{Z}\pi)$ is the i th homotopy group of the pseudo-isotopy space of a PL manifold whose fundamental group is π . In addition to applications to topology, K -theory has applications in number theory and is a powerful tool for the study of C^* -algebras. As K -theory becomes more established and more widely known, more applications will, no doubt, be found.

At some schools algebraic K -theory has become a basic part of the graduate algebra curriculum. Lower K -theory (K_0, K_1, K_2) is more accessible to the beginning algebraist than many other topics which are more often taught (eg. class field theory). Furthermore, many of the standard techniques of lower algebraic K -theory are applicable in other branches of algebra. Therefore, K -theory is a very reasonable subject for a second year graduate course and should be considered for such at most graduate schools. Until now no appropriate textbook has been available. Atiyah's book [1] on K -theory is inappropriate because it covers mainly topological K -theory and uses topological methods. Bass's book [2] is a reference book for specialists which is inaccessible to the beginner and covers only K_0 and K_1 . Milnor's book [3] is readable, but assumes a great deal of mathematics and mathematical sophistication. Furthermore, many proofs are left to the reader. The book under review is a textbook which is developed from a course given at King's College. It includes some standard topics in algebra (eg. modules, tensor products, localization) as they are needed. The book presents K_0, K_1 , and K_2 , covering selected topics beyond the definitions. It does include the category-theoretic definitions as well as the strictly algebraic ones. The variety of abelian groups which arise as K -groups is not shown since the examples are restricted to fields, division rings, local rings and $\mathbf{Z}[\sqrt{-d}]$. The format of the book is essentially theorem-proof which sometimes leaves the reader wondering where the book is going. There is, however, more motivation and historical discussion in the chapters which cover K_0 than in the ones on K_1 and K_2 . The proofs, for the most part, are given in great detail, often tediously so for the mathematically mature reader, but useful to the beginning student.

The book is remarkably free of misprints and has a very helpful list of notations which includes the page on which the notation is introduced. Another helpful practice is that in the later chapters, if a result is used from the beginning of the book, a page number for the reference is often given. Furthermore, when the author is going to abuse notation, he not only points

out what he is doing but also says why he is doing it. The exercises are usually interesting but sometimes routine.

This book is accessible to all mathematicians and any graduate student who has had a course in abstract algebra. For the latter, some of the references (eg. references to algebraic varieties) will be obscure, but not to the point of making the book unreadable. Furthermore, the book presents clearly many of the standard techniques of abstract algebra in a fashion which is helpful to a second year graduate student. The exposition is dry but clear, so that the book could be used in a reading course. It is not a book to read quickly to get the flavor of K -theory, but rather a book to be worked through to gain a feel for the subject.

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Emmy Noether, a tribute to her life and work, edited by James W. Brewer and Martha K. Smith, Pure and Applied Mathematics: A Series of Textbooks and Monographs, vol. 69, Dekker, New York and Basel, 1981, x + 180 pp., \$14.75.

The centenary of E. Noether's birth gave rise to two English language tributes very different in intent, background and execution. The first one was written in German in 1968; it was translated in English only in 1980. Its author, a teacher of mathematics living in Vienna, was motivated by a lecture of Professor E. Hlawka (Vienna University) on the development of mathematics in the last hundred years, as she states in the preface of her book. She traced very carefully the available documentary evidence of the life path and the scholarly development of E. Noether in context (see review of *Emmy Noether, 1882–1935*, by Auguste Dick, in Bull. Amer. Math. Soc. **6** (1982), 224–230). The second one is specially written for the occasion by mathematicians who encountered and applied Noetherian modes of thought in their work. The desire to describe the influence of E. Noether on contemporary research brought forth a number of articles (see 6–10 of the book under review) containing semipopular expositions of *Noether's mathematics* by five specialists (R. G. Swan, E. J. McShane, R. Gilmer, T. Y. Lam, A. Fröhlich). A biography entitled: *Emmy Noether and her influence*, by Clark Kimberling (pp. 1–64) followed by personal recollections of Saunders Mac Lane (pp. 65–78) and Olga Taussky (pp. 79–92) relating to E. Noether and new translations of