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The origins of Cauchy's rigorous calculus, by Judith V. Grabiner, MIT Press, Cambridge, 1981, viii + 252 pp., \$30.00.

Équations différentielles ordinaires: Ordinary differential equations, by Augustin Louis Cauchy, Études Vivantes, Paris, and Johnson Reprint Corporation, New York, 1981, lviii + 146 pp., \$24.50.

Calculus in 1800 was in a curious state. There was no doubt that it was correct. Mathematicians of sufficient skill and insight had been successful with it for a century. Yet no one could explain clearly why it worked. To be sure, experts would probably have agreed that some notion of "limit" lay behind derivatives, and of course integrals were defined as antiderivatives and thus raised no separate questions. But the discussions of the foundational issues had been desultory and inconclusive. Students by and large were not instructed in calculus, they were initiated into it. If they were gifted with the right insight, practice would then give them an intuitive feeling for the right results. The motto of the period, attributed (perhaps wrongly) to D'Alembert, was *Allez en avant, et la foi vous viendra*: Go forward, and faith will come to you.

Still, there was a nagging feeling that something should be done. There were occasional disagreements, like that over the vibrating string, that were hard to bring to a clear resolution. Besides, mathematicians still remembered the tradition of proof that was their proud inheritance from the Greeks. To establish something "in the style of geometry" was a byword for establishing it beyond doubt. Particularly galling was the fact that Archimedes had established some "calculus" results in exactly that style. It seemed in fact that every single area value or tangent slope computed by calculus could be similarly justified. But no one wanted to do such justifications, because they were long, tedious, and (worst of all) apparently unrelated to the intuition behind the calculus. Lagrange had been concerned with justifying calculus for over twenty years, but his major efforts had rested on the formal use of power series and were not satisfactory.

Then came Cauchy. It is hardly enough to say merely that he *solved* the problems; he showed that there *weren't* any problems. Seldom has there been such good reason to say, with Boileau,

Ce que l'on conçoit bien s'énonce clairement,
 Et les mots pour le dire arrivent aisément.

For Cauchy was not at all the type of scholar who ponders and polishes his work for years. Throughout his career he wrote almost two papers a month. His last submission to the Académie des Sciences, less than three weeks before he died, ends with the words, "I shall explain this at greater length in a memoir to follow." Called upon to lecture on calculus, he merely presented the prescribed topics as best he could. But his best was so illuminating that the

ideas stood out with quite unaccustomed clarity. He did not really establish new foundations; he swept away all the dust to reveal the whole edifice of calculus already standing on bedrock.

This is the accomplishment treated in the new book by Grabiner. Its outlines, of course, have long been familiar; we hardly hold our breath in suspense wondering how it will all come out. But her book nonetheless introduces a valuable new approach. Starting from something basic, like Cauchy's definition of the derivative, one can trace its historical antecedents in two ways. First, one can ask what previous attempts had been made to define derivatives. This has been done by many authors. But Grabiner also asks where else Cauchy might have seen ideas like those he used in his definitions. The first question leads, by and large, only to various minor works by authors like James Jurin, Benjamin Robins, Simon L'Huilier, and Lazare Carnot. The second, as she shows, leads us rather to people like Euler, D'Alembert, Ampère, and Poisson. Slightly oversimplifying her conclusions, we may say that, alongside the formal machinery of calculus, certain approximation techniques had been developed in various special situations. Occasionally (though not often) these were accompanied by error estimates. Cauchy's great insight was that he could view these as fundamental. Using them as definitions (and developing them into proofs), he could reach all the formal results while using nothing more "transcendental" than inequalities. This is just what Archimedes had done in his special cases; drawing on a century's accumulated experience in manipulating inequalities, Cauchy could carry out the arguments once and for all.

A secondary theme in Grabiner's book is the importance of Lagrange. Grabiner stresses three aspects of this. First, in the later 1700's Lagrange had been by far the most important mathematician seriously concerned with the foundations of calculus, and she can document how much his work increased other people's awareness of the problems. Second, he more than his predecessors was concerned with formulating general approximation techniques and giving error estimates for them. Finally, in a number of places Lagrange got at least part way toward the right understanding. For instance, consider the condition that $f(x + h) = f(x) + f'(x)h + hR$ with $R \rightarrow 0$ as $h \rightarrow 0$. Lagrange did not define the derivative in this way; but he stated the property early in his treatment and based several later arguments on it. Also, by emphasizing that differentiation applied to functions (and thus could be iterated), he eliminated all the special problems raised by "higher-order infinitesimals". The next time you run through higher derivatives in less than one meeting of the freshman calculus course, pause to murmur a short blessing on the name of Lagrange.

As one reads the book, one point that stands out is that Cauchy's definitions actually play a different role from those of his predecessors. In the eighteenth century, people gave definitions of limits; many of them, echoing a phrase of Newton's, said that the limit was a value to which the variable came arbitrarily close, without ever exceeding it. In practice, of course, analysts were perfectly familiar with cases where the variable oscillated around its limit. But the "definition" was meant only as a sort of guide to the intuition, like "A point is position without extension" at the start of Euclid. Once the calculus formalism

had been grasped intuitively, the definitions were never again mentioned, so the later exceptions never mattered. Cauchy's definitions, on the other hand, are not intuitive suggestions; they are precise formulations of what he means. Consequently, they can be used explicitly whenever necessary. And it is just this explicit use of inequalities that separates analysis today from the calculus of the eighteenth century. At the risk of sounding paradoxical, one might say that clarifying the concepts of calculus was in itself the smaller part of Cauchy's accomplishment in clarifying the concepts of calculus. In laying bare their roots, he also cleared the ground for all the flourishing growth of modern analysis.

Another point that stands out is the importance of advanced studies in clarifying the basics. The influence of approximation studies has already been mentioned. For another example, consider integration. For most people in the eighteenth century, integrals were defined as antiderivatives; Cauchy made definite integrals basic. Why? One reason, of course, was that existence proofs could then be given even when no antiderivative was apparent. But when all functions in sight had had power series expansions, antiderivatives always *were* apparent. The existence question really became important only when people wanted to claim that they could expand "arbitrary" periodic functions in Fourier series. A second reason was that Cauchy and others were beginning to explore functions of a complex variable, and a naive use of them in integration gave some quite puzzling results. It became possible to attach a value (indeed, many values) to $\log(-1)$; but no possible choice could make $\log(1) - \log(-1)$ real, and what then did it have to do with $\int_{-1}^1 dx/x$? Yet a third reason, I suspect—one that Grabiner overlooks—is that around this time Cauchy was doing quite a lot of work on differential equations. Here again he emphasized the initial value problem, where most previous authors had dealt with formal solutions containing indeterminate constants. In this case he could get precise theorems that were not apparent when people had to be concerned with "special solutions" that might or might not be particular cases of the "general solutions". Even more is this true in partial differential equations, where we still say "Cauchy problem" when we mean the initial value problem. It is worth remembering that Cauchy had already discovered the method of characteristics before he gave the lectures in the *Cours d'analyse*.

This last thought was suggested by the second book under review, the heart of which is a photoreproduction of Cauchy's 1823–1824 lecture notes on ordinary differential equations. (Despite the bilingual title, by the way, all but one page of the book is in French.) In structure, in notation, sometimes even in wording the central existence proof is completely parallel to that for definite integrals in the earlier lectures. Again Cauchy takes an approximation method, one mentioned incidentally once or twice by his predecessors, and turns it into a major proof. (There is also the same oversight as for definite integrals: continuity on closed bounded intervals is tacitly assumed to be uniform.) As a whole the lectures are clear and pleasant to read, better perhaps than some books on the market today. Cauchy even gives a preliminary treatment of $y' = F(x, y)$ with the extra assumption that F is bounded on an infinite strip $a \leq x \leq b$; only then does he go on to the more complicated estimates where F

is merely continuous. It is interesting to see that this useful pedagogical device goes back to the founder of the theory.

The notes reproduced here are all set in print; it seems very likely that some of them are Cauchy's own proof sheets. But they break off as Cauchy begins to extend the existence theorem to systems, and they were never published. Cauchy's existence proof was made known in a book by Moigno in 1840, but these notes remained totally unknown until they were discovered a few years ago by Christian Gilain. His fifty-page introduction discusses how he found them, what they contain, and why Cauchy never published them. This discussion is interesting in several ways. First of all, Gilain knew what to look for because he had located the official course outlines for the calculus sequence at the École Polytechnique in the 1820s. These also revealed that the three sets of notes published by Cauchy, the *Cours d'analyse* (1821), the *Résumé des leçons. . . sur le calcul infinitésimal* (1823), and the *Leçons sur les applications du calcul infinitésimal à la géométrie* (1826), were not (as previously thought) separate incomplete treatments but rather a single sequence, following exactly the topics prescribed for the first year course.

These notes all came into existence because in 1820 the École Polytechnique prescribed that if a lecturer covered material not in the standard texts, he should write up notes on it for the students, who would be taking exams on it the next September. The surviving printer's dates on the differential equations notes show that with them Cauchy fell behind. This may not have been totally his fault, for the notes were printed by the same office that did all the rest of the government printing, and about this time they fell behind in several sets of notes because of work stemming from some financial laws. At any rate, the printing was probably dropped once it was too late to help the students in their exams.

Gilain also has a reasonable explanation why Cauchy did not publish more on this method in his later career. In the 1830s, he discovered the "majorant" arguments to establish convergence of power series solutions, and he probably thought that these were a more satisfactory approach. For one thing, they gave the solutions in much more traditional form. Still better, they applied to functions of a complex variable. Indeed, in their original form they were tied in with the "Cauchy estimates" for $f^{(n)}(0)$ in terms of the values of $f(z)$ on a circle, and so they applied only to functions extended to a complex variable. It is good to be reminded how much the study of ordinary differential equations focused on complex functions in the middle of the nineteenth century. In fact, Cauchy's original, real existence theorem remained almost unnoticed until it was rediscovered (with some improvements) by Lipschitz fifty years later (1876). Even then, Jordan, in the first edition of his famous *Cours d'analyse*, could devote an entire volume to differential equations (1886) without mentioning this theorem. Only the work of Poincaré brought the focus back to real-variable problems.

One further question remains. Cauchy gave the same differential equations course again two years later; why weren't notes for the complete course printed then? Gilain's answer to this forces some revision of common ideas about the position of mathematics in France at the time. Felix Klein, for instance, in his

fascinating old history of mathematics in the nineteenth century, had cited Cauchy's lectures as evidence of the "unusually high requirements on the purely mathematical side that were set as a basis" for the practical instruction in the École Polytechnique. But this turns out not to be true. In fact, there were complaints about Cauchy's teaching—and my, but they do sound familiar! *Five* lectures on the generalities of integration? "That might be all right in the Faculty of Sciences," said a physicist, "but it is not appropriate in the École Polytechnique, where the students are pressed for time." By 1825, when Cauchy repeated the course on differential equations, the Ministry of Education had been persuaded to decree officially that lecturers should stick to the syllabus officially established. Officially, Cauchy agreed: the minutes for November, 1825 say "M. Cauchy announces that, to conform to the wishes of the Council, he will no longer strive, as he has up to now, to give perfectly rigorous proofs." But in fact he did not change. The minutes a year later record that "M. Cauchy has presented only lecture notes that could not satisfy the commission, and thus far it has been impossible to make him... carry out the decision of the Minister." In other words, his notes that year were not considered fit to print.

Cauchy was always a man of prickly principles. Loyal to the old Bourbon regime, he abandoned his positions rather than swear an oath of allegiance to Louis Philippe after the 1830 revolution. (In 1838 he resumed activity in the Académie, which was exempt, but still refused all positions requiring the oath.) He had even less liking for the republican government set up in 1848, but he immediately resumed his position at the Sorbonne—because an oath was no longer required. Personal details like this are usually mere diversions in the history of mathematics, but in this particular case they seem to be important. As several authors (including Grabiner) have pointed out, Cauchy was not the only mathematician to lecture on calculus at the École Polytechnique. Ampère, Poisson, and others did so at about the same time. But Cauchy was stubborn. He would no more choose to give a false proof than to swear a false oath; he would deliver *his* lectures *his* way. And it seems that his stubbornness as well as his genius helped to give us the *Cours d'analyse*.

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Numerical methods for stiff equations and singular perturbation problems, by Willard L. Miranker, *Mathematics and its Applications*, vol. 5, D. Reidel Publishing Company, Dordrecht, Holland; Boston, U.S.A.; London, England, 1981, xiii + 202 pp., \$29.95.

Numerical analysis and perturbation theory are two principal approaches to the problems of applied mathematics. It is a little surprising that there has not been more interaction between these approaches. In my opinion this is because the goals and the problem classes are rather different. At the risk of gross