Let $M_n$ be the set of $n \times n$ matrices over the algebraically closed field $k$, $G_n$ the general linear group in $M_n$, $M_{n,m} = M_n \times \cdots \times M_n (m+1 \text{ times})$. $G_n$ acts naturally on $M_{n,m}$ by the conjugation $T M_{n,m} T^{-1}$. For $\alpha = (A_0, \ldots, A_m) \in M_{n,m}$ denote by $\text{orb}(\alpha)$ the orbit of $\alpha$ in $M_{n,m}$.

$$\text{orb}(\alpha) = \{ \beta \in M_{n,m}, \beta = T\alpha T^{-1} = (TA_0 T^{-1}, \ldots, T A_m T^{-1}), T \in GL_n \}.$$  

It is a well-known problem to classify $\text{orb}(\alpha)$ for $m \geq 1$. See for example [2]. Rosenlicht in [3] outlined a general classification based on the ideas of algebraic geometry. The classification consists of a finite number of steps. In each step we get an algebraic irreducible variety $V$ in $M_{n,m}$ which is invariant, that is $TVT^{-1} = V$ for all $T \in G_n$. Then, we consider $k(V)^G$—the field of rational functions on $V$ which are invariant, i.e. these functions are constant on $\text{orb}(\alpha)$. It follows that $k(V)^G$ is finitely generated, let us say by $\chi_1, \ldots, \chi_j$. Then there exists locally closed algebraic invariant set $V^0$ in $V$ such that for any $\alpha \in V^0 \chi_1, \ldots, \chi_j$ are well defined on $\text{orb}(\alpha)$ and the values of $\chi_k, k = 1, \ldots, j$, on $\text{orb}(\alpha)$ determine this orbit uniquely in $V^0$.

The purpose of this announcement is to describe explicitly the open invariant varieties $V^0$ together with the invariant rational functions $\varphi_1, \ldots, \varphi_k$ defined on $V^0$ such that the values of $\varphi_1, \ldots, \varphi_k$ on $\text{orb}(\alpha)$ determine a finite number of orbits. We also describe some results on orbits in $S_{n,m} = S_n \times \cdots \times S_n (m+1 \text{ times})$ ($S_n$ is the set of $n \times n$ complex symmetric matrices) under the action of $O_n$-complex orthogonal group in $M_n$.

For $\alpha = (A_0, \ldots, A_m)$, $\beta = (B_0, \ldots, B_m)$ let $\text{adj}(\alpha, \beta): M_n \to M_{n,m}$ be a linear operator given by $\text{adj}(\alpha, \beta)(X) = (A_0 X - X B_0, \ldots, A_m X - X B_m)$.

We identify $\text{adj}(\alpha, \alpha)$ with $\text{adj}(\alpha)$. Let $r(\alpha, \beta)$ and $r(\alpha)$ be the ranks of $\text{adj}(\alpha, \beta)$ and $\text{adj}(\alpha)$ respectively. Then $r(\alpha)$ is the first discrete invariant of $\text{orb}(\alpha)$ and it gives the dimension of the manifold $\text{orb}(\alpha)$. Suppose that $\beta \in \text{orb}(\alpha)$. Then one easily shows that $r(\alpha, \beta) = r(\alpha)$. Fix $\alpha$ and consider all $\xi \in M_{n,m}$ which satisfy the inequality

$$(1) \quad X(\alpha) = \{ \xi, r(\alpha, \xi) \leq r, \xi = (X_0, \ldots, X_m) \in M_{n,m} \}.$$  

The set $X(\alpha)$ is an algebraic set in $M_{n,m}$ which can be given by

$$N(r) = \left( \begin{array}{cc} n^2 & (m+1) \\ r+1 & r+1 \end{array} \right) \text{polynomial equations}.$$  

Received by the editors December 22, 1981.

1980 Mathematics Subject Classification. Primary 14D25, 14L30, 15A21.

Key words and phrases. Simultaneous similarity, invariant functions, symmetric matrices.

© 1983 American Mathematical Society
Indeed, in tensor notation, \( \text{adj}(\alpha, \xi) \) is represented as the following matrix

\[
\text{adj}(\alpha, \xi) = (I \otimes A_0 - X_0^t \otimes I, \ldots, I \otimes A_m - X_m^t \otimes I)
\]

where \( X^t \) denotes the transposed matrix of \( X \). Let \( f_i(\alpha, \xi), i = 1, \ldots, p = N(\tau) \) be all \((r+1) \times (r+1)\) minors of \( \text{adj}(\alpha, \xi) \). Then (1) is given by the equations

\[
f_i(\alpha, \xi) = 0, \quad i = 1, \ldots, N(\tau).
\]

Let \( \mathcal{W}_r \) be a linear space of all polynomials \( p(\xi) \) in the \((m + 1)n^2\) entries of \( X_0, \ldots, X_m \) of degree \( d \leq r + 1 \). Denote by \( u_1 = u_1(\xi), \ldots, u_{s(r)} = u_{s(r)}(\xi) \) the standard basis in \( \mathcal{W}_r \). Then

\[
f_i(\alpha, \xi) = \sum_{j=1}^{s(r)} \pi_{ij}^{(r)}(\alpha)u_j(\xi), \quad i = 1, \ldots, N(\tau).
\]

Put \( \pi^{(r)}(\alpha) = (\pi_{ij}^{(r)}(\alpha)), i = 1, \ldots, N(\tau), j = 1, \ldots, s(r) \). Then

\[
\rho(\alpha) = \text{rank } \pi^{(r)(\alpha)}(\alpha)
\]

is the second discrete invariant of \( \text{orb}(\alpha) \). Define

\[
V_{r,\rho}^0 = \{ \alpha, \alpha \in M_{n,m}, \text{rank } \text{adj}(\alpha) = r, \text{rank } \pi^{(r)}(\alpha) = \rho \}.
\]

Then \( V_{r,\rho}^0 \) is an open algebraic set in \( M_{n,m} \). (It may be empty for some choices of \( r \) and \( \rho \).)

Finally, we recall that two \( p \times q \) rectangular matrices \( A \) and \( B \) are row equivalent \( (A \sim B) \) if there exists a nonsingular matrix \( Q \) such that \( B = QA \). Any \( p \times q \) matrix \( A \) can be brought to the unique row-echelon form \( E \) using the elementary row operations. \( E = (e_{ij}) \) is characterized by \( 1 < p_1 < \cdots < p_\rho \leq q \), \( \rho = \text{rank } E \), since \( e_{ip_i} = 1, e_{ip_j} = e_{iq} = 0 \) for \( j < i, q < p_i \) and \( i > \rho \). The integers \( p_1, \ldots, p_\rho \) are called the discrete invariants of \( E \) and the entries \( e_{ij}, p_i < j, j \neq p_{i+1}, \ldots, p_\rho, \) for \( i = 1, \ldots, \rho \) are called the continuous invariants of \( E \). Once \( p_1, \ldots, p_\rho \) are specified these invariants are given as well-determined rational functions of entries of \( E \).

**Theorem 1.** Assume that \( V_{r,\rho}^0 \) is nonempty. Let \( \alpha, \beta \in V_{r,\rho}^0 \). If \( \beta \in \text{orb}(\alpha) \) then \( \pi^{(r)}(\alpha) \sim \pi^{(r)}(\beta) \). Moreover, there are at most \( \kappa = \tau(n^2 - r)(mn^2 + n^2 - r) \) distinct orbits \( \text{orb}(\alpha_1), \ldots, \text{orb}(\alpha_k) \) such that \( \pi^{(r)}(\alpha_1), \ldots, \pi^{(r)}(\alpha_k) \) have the same row-echelon form.

**Sketch of the Proof.** We first note that if \( \beta \in \text{orb}(\alpha) \) then \( \pi^{(r)}(\alpha) \sim \pi^{(r)}(\beta) \). Indeed, since \( \beta = T\alpha T^{-1} \) the tensor representation of \( \text{adj}(\alpha, \xi) \) yields

\[
\text{adj}(\beta, \xi) = T_1 \text{adj}(\alpha, \xi) \text{diag}\{T_1^{-1}, \ldots, T_1^{-1}\}, \quad T_1 = I \otimes T.
\]

The Cauchy-Binet formula implies that any minor of \( \text{adj}(\beta, \xi) \) is a linear combination of all \((r+1) \times (r+1)\) minors of \( \text{adj}(\alpha, \xi) \) and the coefficients in this dependence are functions of \( T \), i.e. independent of \( \xi \) ! Whence the subspace spanned by the rows of \( \pi^{(r)}(\alpha) \) contains the rows of \( \pi^{(r)}(\beta) \). Interchanging the roles of \( \alpha \) and \( \beta \) we get \( \pi^{(r)}(\alpha) \sim \pi^{(r)}(\beta) \). Fix \( \alpha \). We then show the existence of a neighborhood \( D(\alpha) \) such that the conditions \( \beta \in D(\alpha) \) and \( \pi^{(r)}(\alpha) \sim \pi^{(r)}(\beta) \) imply that \( \beta \in \text{orb}(\alpha) \). For that, in the matrix \( \text{adj}(\alpha) \) pick up a nonzero \( r \times r \) minor. We then consider the corresponding \( r \) linear equations out of \((m + 1)n^2\) equations \( A_iX - XA_i = 0, \quad i = 0, \ldots, m \). This \( r \)-system has \( n^2 - r \) free parameters \( x_{ij}, (i, j) \in \mathcal{A}(X = (x_{ij})) \). Since \( X = I \) is a solution, the above
system has the unique solution \( X = I \) whose free parameters are given by \( x_{ij} = \delta_{ij}, (i,j) \notin A \). Consider the same \( r \)-equations in a more general system
\[
A_i X - X B_i = 0, \quad i = 0, \ldots, m.
\]
Thus, there exists a neighborhood \( D(\alpha) \) of \( \alpha \) in \( M_{n,m} \) such that for any \( \beta \in D(\alpha) \) the above \( r \)-system is linearly independent and has the unique solution \( X(\alpha, \beta) \) with \( \det X(\alpha, \beta) \neq 0 \). Suppose that \( \pi^{(r)}(\alpha) \sim \pi^{(r)}(\beta) \). So each \( (r+1) \times (r+1) \) minor of \( \text{adj}(\beta, \alpha) \) is a linear combination of all \( (r+1) \times (r+1) \) minors of \( \text{adj}(\alpha, \alpha) \) which are equal to zero! So rank \( \text{adj}(\alpha, \beta) = \text{rank} \text{adj}(\beta, \alpha) \leq r \). If in addition \( \beta \in D(\alpha) \) then rank \( \text{adj}(\alpha, \beta) = r \) and the matrix \( X(\alpha, \beta) \) must satisfy all \( (m+1)n^2 \) equalities \( A_i X - X B_i = 0, \quad i = 0, \ldots, m \). So \( \beta \in \text{orb}(\alpha) \). Consider finally the variety \( X = X(\alpha) \). Let \( X = \bigcup_{i=1}^{k} X_i \) be the decomposition of \( X \) into irreducible components. To this end we show that each \( X_i \) contains at most one orbit. Assume that \( \alpha \in X_1 \) and let \( X_1 \) be the open manifold of all regular points of \( X \). The above arguments prove that \( D(\alpha) \cap X_1 \subset \text{orb}(\alpha) \). On the other hand \( \text{orb}(\alpha) \subset X_1 \). As \( X_1 \) and \( \text{orb}(\alpha) \) are connected we deduce that \( \text{orb}(\alpha) = X_1 \). A simple degree argument shows that \( k \leq \kappa \). Therefore we have at most \( \kappa \) distinct orbits.

Let \( 1 \leq p_1 < p_2 < \cdots < p_\rho \leq q = s(r) \). Let \( V_{\pi, p_1, \ldots, p_\rho}^0 \) be the set of all \( \alpha \in V_{\pi, p_1, \ldots, p_\rho}^0 \) whose row-echelon form of \( \pi^{(r)}(\alpha) \) has the discrete invariants \( p_1, \ldots, p_\rho \). Then the entries \( e_{ij}, p_i < j, \quad j \neq p_{i+1}, \ldots, p_\rho, \quad i = 1, \ldots, \rho \), in the row-echelon form the invariant rational functions which determine the \( \text{orb}(\alpha) \) up to \( \kappa \) orbits at most. In fact, we conjecture that if \( \alpha \) and \( \beta \) lie in the same connected component of \( V_{\pi, p_1, \ldots, p_\rho}^0 \) and \( \pi^{(r)}(\alpha) \sim \pi^{(r)}(\beta) \) then \( \text{orb}(\alpha) = \text{orb}(\beta) \).

For \( \alpha \in S_{n,m} \) let \( \text{sorb}(\alpha) = \{ \beta, \beta = T \alpha T^{-1}, \quad T \in O_n \}, \quad \alpha(z) = \sum_{i=0}^{m} A_i z^i \), where \( z \in C \) (the field of complex numbers). Let \( p(\lambda, z) = \det(\lambda I - \alpha(z)) \) be the characteristic polynomial of \( \alpha \). Clearly \( p(\lambda, z) \) is invariant on \( \text{sorb}(\alpha) \) or \( \text{orb}(\alpha) \). It can be shown that for most \( \alpha \in S_{n,m} \) the equation \( p(\lambda, z) = 0 \) \( (\alpha \text{ is fixed}) \) will have \( n \) distinct \( \lambda \) roots for all except a finite number of \( z \), possibly \( z = \infty \) \( (p(\lambda, \infty) = \det(\lambda I - A_m)) \) and at those exceptional points the equation \( p(\lambda, z) = 0 \) will not have triple roots. We call such \( \alpha \) and corresponding \( p(\lambda, z) \) simple.

**Theorem 2.** There are at most \( 2^{(n-1)(m-n-1)} \) distinct \( \text{sorb}(\alpha_1), \ldots, \text{sorb}(\alpha_k) \) such that all these orbits have the same simple characteristic polynomial.

We conjecture that if \( A_0, \ldots, A_m \) are real symmetric then \( \text{sorb}(\alpha) \) is determined by its characteristic polynomial up to a finite number of orbits.

The detailed results are given in [1].

**References**


Institute of Mathematics, Hebrew University, Jerusalem, Israel

Department of Mathematics, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201