BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 8, Number 2, March 1983

EXISTENCE THEOREMS FOR GENERALIZED KLEIN-GORDON EQUATIONS

BY EZZAT S. NOUSSAIR AND CHARLES A. SWANSON[1]

The semilinear elliptic partial differential equation

$$(1) \qquad Lu = f(x,u), \qquad x \in \Omega,$$

is to be considered in smooth unbounded domains $\Omega \subseteq R^N$, $N \geq 2$, where

$$(2) \qquad Lu = -\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_j u] + b(x)u, \qquad x \in \Omega,$$

$D_i = \partial/\partial x_i$, $i = 1,\ldots,N$; each $a_{ij} \in C^{1+\alpha}_{\text{loc}}(\Omega)$, $b \in C^{\alpha}_{\text{loc}}(\Omega)$, $0 < \alpha < 1$; $b(x) \geq b_0 > 0$ for all $x \in \overline{\Omega}$, L is uniformly elliptic in Ω, and $f(x,u)$ satisfies all the conditions in either list (F) or list (F′) below. Our main objective is to prove the existence of a positive solution $u(x)$ of (1) in Ω satisfying the boundary condition $u(x) = 0$ on $\partial\Omega$ (void if $\Omega = R^N$), and to obtain asymptotic estimates as $|x| \to \infty$.

The physical importance of the Klein-Gordon prototype

$$(3) \qquad -\Delta u + b(x)u = \delta[p(x)u^\gamma - q(x)u^\beta], \qquad x \in \Omega,$$

arises in particular from nonlinear field theory; the existence of solitary waves and asymptotic behavior as $|x| \to \infty$ follow from our theorems. It is assumed in (3) that p and q are nonnegative, bounded, and locally Hölder continuous in Ω, $1 < \gamma < \beta$, and $\delta = \pm 1$. The Hypotheses (F′) below are all satisfied if $\delta = +1$ and p/q is bounded and bounded away from zero in Ω. Hypotheses (F) are all satisfied if $\delta = -1$, $\beta < (N+2)/(N-2)$, $N \geq 3$, and $q(x) > 0$.

HYPOTHESES F (UNBOUNDED NONLINEARITY)

(f_1) $f \in C^{\alpha}_{\text{loc}}(\Omega \times R)$ and $f(x,t)$ is locally Lipschitz continuous with respect to t for all $x \in \Omega$.

(f_2) There exist positive constants $s_i > 1$ and nonnegative, bounded continuous functions $f_i \in L^2\Omega$, $i = 1,\ldots,I$, such that

$$|f(x,t)| \leq \sum_{i=1}^{I} f_i(x)|t|^{s_i}, \qquad x \in \overline{\Omega},\ t \in R,$$

where each $s_i < (N+2)/(N-2)$ if $N \geq 3$.

(f_3) $f(x,t)/t \to +\infty$ as $t \to +\infty$ locally uniformly in Ω.

Received by the editors May 18, 1982 and, in revised form, September 3, 1982.
1980 *Mathematics Subject Classification*. Primary 35J65; Secondary 35B40.
[1]Support of NSERC (Canada) is gratefully acknowledged.

0273-0979/82/0000-1201/$01.75

(f_4) There exists a positive constant ϵ such that $(2+\epsilon)F(x,t) \leq tf(x,t)$ for all $t \geq 0$, $x \in \Omega$, where $F(x,t) = \int_0^t f(x,\tau)d\tau$.

HYPOTHESES F′ (BOUNDED NONLINEARITY)

(f'_1) = (f_1); (f'_2) $f(x,0) = 0$ for all $x \in \Omega$.

(f'_3) There exists a positive number T such that $f(x,t) < 0$ for all $t > T$ and for all $x \in \Omega$.

(f'_4) There exists $x_0 \in \Omega$ and $T_0 \in [0,T)$ such that $F(x_0,T_0) > 0$.

(f'_5) For every bounded domain $M \subset \Omega$ and for every $t_0 > 0$, there corresponds a positive constant $K = K(M,t_0)$ such that $f(x,t) + Kt$ is a nondecreasing function of t on $0 \leq t \leq t_0$ for each fixed $x \in \overline{M}$.

The unbounded domain Ω in (1) is allowed to have the general form $\bigcup_{n=1}^{\infty} \Omega_n$, where $\{\Omega_n\}$ is a sequence of smooth bounded domains with $\overline{\Omega}_n \subset \overline{\Omega}_{n+1} \subset \overline{\Omega}$ for $n = 1, 2, \ldots$. For example, Ω can be an exterior domain, cylindrical or conical domain, or the entire space R^N.

THEOREM 1. *Suppose that Hypotheses* (F) *hold and that each a_{ij} and $D_i a_{ij}$ in* (2) *is bounded in $\Omega \cup \partial\Omega$. Then equation* (1) *has a positive bounded solution $u(x)$ in Ω satisfying $u(x) = 0$ identically on $\partial\Omega$ such that $u(x) \to 0$ and $|\nabla u(x)| \to 0$ as $|x| \to \infty$ uniformly in Ω. In the case that $\Omega = R^N$,* (1) *has a positive bounded solution $u(x)$ throughout R^N with this asymptotic behavior at* ∞.

Specializing to the Schrödinger operator $L = -\Delta + b(x)$, $x \in \Omega$, we prove that the positive solution $u(x)$ in Theorem 1 satisfies

$$\overline{u}(|x|) \leq C\exp(-k|x|), \qquad x \in \Omega$$

for some positive constants C and k, where $\overline{u}(t)$ denotes the spherical mean square of $u(x)$ over the $(N-1)$-sphere of radius t. Sharper estimates arise in the Klein-Gordon case (3) *if $b(x) = b$ is a positive constant.*

In the case of bounded nonlinearities, we consider boundary value problems of the type

$$\begin{cases} Lu = \lambda f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

in an *exterior domain* Ω, or in the entire space R^N with the boundary condition deleted, where λ is a positive constant.

THEOREM 2. *If Hypotheses* (F′) *hold, there exists a positive number λ^* such that the boundary value problem* (4) *has a bounded positive solution $u(x,\lambda)$ in $\Omega \cup \partial\Omega$ for all $\lambda \geq \lambda^*$. The same is true in the case that $\Omega = R^N$, where the boundary condition in* (4) *is now deleted.*

The theorems below concern the specialization

$$\begin{cases} -\Delta u + b(x)u = \lambda f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

UNIQUENESS THEOREM 3. *Suppose in addition to* (F′) *that* $f(x,t)/t$ *is bounded in* $\Omega\times(0,T]$ *and that* $f(x,t)/t\to 0$ *as* $t\to 0+$ *for all* $x\in\Omega$*. Then there exists a positive number* λ_* *such that the only nonnegative bounded solution* $u(x,\lambda)$ *of* (5) *is identically zero in* Ω *for* $0<\lambda\le\lambda_*$.

THEOREM 4. *Suppose in addition to* (F′) *that* $f(x,t)/t\to 0$ *as* $|x|\to\infty$ *uniformly in* $0<t\le T$*. Then there exists* $\lambda^*>0$ *such that, for all* $\lambda\ge\lambda^*$*, the boundary value problem* (5) *has a positive solution* $u(x,\lambda)$ *in* Ω *satisfying* $u(x,\lambda)\le C(\lambda)\exp[-\sqrt{b_0/2}|x|]$ *for some positive constant* $C(\lambda)$.

We also prove analogous theorems without the uniform positivity hypothesis on $b(x)$, e.g. $b(x)$ can be identically zero, or even have negative values.

To prove Theorem 1, we first construct a sequence of solutions U_n of Dirichlet problems on bounded subdomains Ω_n of Ω, $n=1,2,\dots$, using a variational method of Ambrosetti and Rabinowitz [**1**]. Let u_n denote the extension of U_n to Ω defined to be 0 in $\Omega\backslash\Omega_n$. Using (F) and the variational characterization of U_n we prove that the sequence of Dirichlet norms $||u_n||_{1,2,\Omega}$ is uniformly bounded and uniformly positive. Then *a priori* estimates, embedding theorems, and a "bootstrap procedure" establish the convergence of a subsequence of $\{u_n\}$ locally uniformly in $C^2(\Omega)$ to a solution $u(x)$ of (1) satisfying $u(x)=0$ identically on $\partial\Omega$. Furthermore, these techniques imply that there exists a positive constant C, independent of x, such that both

$$|u(x)|\le C||u||_{1,2,M(x)},\, |\nabla u(x)|\le C||u||_{1,2,M(x)}$$

for all $x\in\Omega$, where $M(x)$ denotes a bounded domain for all $x\in\overline{\Omega}$ with volume of $M(x)$ constant. The asymptotic behavior of $u(x)$ stated in Theorem 1 then follows since $u\in W_0^{1,2}(\Omega)$. This and a comparison argument show that the solution is positive throughout Ω and exponentially decaying as $|x|\to\infty$. Theorem 1 extends known results of Berestycki and Lions [**3**], Berestycki, Lions, and Peletier [**4**], Berger [**5**], Berger and Schechter [**6**] and Strauss [**10**] in three directions: general coefficients (i.e. not necessarily constant or radially symmetric), general domains, and problems with boundary conditions.

We prove Theorem 2 by first constructing subsolutions w_n of Dirichlet problems in bounded domains Ω_n, $n=1,2,\dots$, possible for $\lambda\ge\lambda^*>0$ because of a theorem of Rabinowitz [**9**, p. 177] and a new extension result. Then there exists a sequence of solutions u_n of (1) in Ω_n squeezed between w_n and the constant supersolution T by a theorem of Amann [**2**, p. 283], and u_n is extended by the definition $u_n=0$ in $\Omega\backslash\Omega_n$. Following our method in [**8**] we use L^p-estimates, Sobolev embedding, and Schauder estimates to prove that $||u_n||_{C^{2+\alpha}(\overline{M})}$ is uniformly bounded with respect to n for any bounded domain $M\subset\Omega$. Then a compactness argument shows that a subsequence of $\{u_n\}$ converges to a bounded positive solution of (4) for $\lambda\ge\lambda^*$. Theorems 3 and 4 can then be established with the aid of Kato's *a priori* estimates [**7**, p. 415].

REFERENCES

1. A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.
2. H. Amann, *Existence and multiplicity theorems for semilinear elliptic boundary value problems*, Math. Z. **150** (1976), 281–295.
3. H. Berestycki and P. L. Lions, *Une méthode locale pour l'existence de solutions positives de problèmes semi-linéaires elliptiques dans* R^N, J. Analyse Math. **38** (1980), 144–187.
4. H. Berestycki, P. L. Lions and L. A. Peletier, *An ODE approach to the existence of positive solutions for semilinear problems in* R^N, Indiana Univ. Math. J. **30** (1981), 141–157.
5. M. S. Berger, *On the existence and structure of stationary states for a nonlinear Klein-Gordon equation*, J. Funct. Anal. **9** (1972), 249–261.
6. M. S. Berger and M. Schechter, *Embedding theorems and quasi-linear elliptic boundary value problems for unbounded domains*, Trans. Amer. Math. Soc. **172** (1972), 261–278.
7. T. Kato, *Growth properties of solutions of the reduced wave equation with a variable coefficient*, Comm. Pure Appl. Math. **12** (1959), 403–425.
8. E. S. Noussair and C. A. Swanson, *Positive solutions of semilinear Schrödinger equations in exterior domains*, Indiana Univ. Math. J. **28** (1979), 993–1003.
9. P. H. Rabinowitz, *Pairs of positive solutions of nonlinear elliptic partial differential equations*, Indiana Univ. Math. J. **23** (1973), 173–186.
10. W. A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. **55** (1977), 149–162.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, KENSINGTON, NEW SOUTH WALES, AUSTRALIA 2033

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, BRITISH COLUMBIA, CANADA V6T 1Y4