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The representation theory of the symmetric group, by Gordon James and Adalbert Kerber, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley, Reading, Mass., 1981, xxviii + 510 pp., $44.50.

There are many excellent texts on the representation-theory of finite groups, e.g. [**21**] for the 'ordinary' theory (over fields of characteristic 0) and [**6**, **8**, **14**, **39**], for the modular theory. The representation-theory of the symmetric groups S_n cannot, at present, be considered merely a specialization of the more general theory; there is a rich accumulation of concepts and theorems in the theory of S_n, whose analogs for arbitrary finite groups do not exist (or have not yet been found). The results of this special theory, not only have an intrinsic interest and beauty, but have applications in chemistry, physics and areas of mathematics as diverse as algebraic geometry (via flag manifolds, determinantal varieties and the Schubert calculus), classical invariant theory, rings with polynomial identity, multivariate statistics (through the work of A. T. James—*not* an author of the text under review!—and many followers; cf. [**13**, Chapters 12 and 13]) and, of course, combinatorics (in particular, via the Robinson-Schensted correspondence and the Redfield-Pólya enumeration-theory).

Two quite different approaches have given rise to most of this theory of S_n:

On the one hand, Frobenius determined the character-table of S_n utilizing certain symmetric polynomials (then called 'bi-alternants', now usually called 'Schur functions' or 'S-functions') and their known properties (which had been intensively investigated by Jacobi and others in the mid-nineteenth century). His student, Schur, extended this work, using it in his thesis to study the representation-theory of $GL(n)$. This approach studies the representations of S_n in terms of their characters, and in the context of the theory of symmetric polynomials.

On the other hand, Alfred Young, in his work on classical invariant theory, was led to study what he called 'Quantitative Substitutional Analysis' (the common title of the brilliant series of nine papers where he developed this approach to invariant- and representation-theory; cf. [**51**]); this theory involves many fundamental constructions, of which the best-known are the Young standard tableaux and the Young idempotents. Young's presentation is very condensed, and a small portion of his work has been popularized in Rutherford [**43**]; the bulk of Young's profound work has not yet been fully understood, in the reviewer's opinion. As developed also by Specht [**45**] and Garnir [**16**], this second approach deals with representations of S_n by constructing explicit S_n-modules, rather than dealing primarily with the associated characters.

(To do full justice to the role played by $GL(n)$ in the theory of S_n, which cannot be done in this review, would require discussion of a third important approach to these matters, based on Lie-algebraic techniques. Cf. also [**5**, **7**, **9**, **19**, **20**, **25**, **48**, **49**, and **52**] for yet other approaches, based on λ-rings, Hopf-algebraic techniques, shape-algebras, Turnbull's double standard tableaux, etc; it remains to be seen which (if any) of these more recent viewpoints will lead to significant new results.)

The text under review is one of a number of current successors to Robinson's classic path-breaking text, *Representation theory of the symmetric group* [**42**]. The James-Kerber text has an introduction by Robinson, in which he courteously praises the merits of their modern treatment (in particular acknowledging that the Decomposition Matrix tables in the James-Kerber text correct some errors in the corresponding tables in the earlier text) and wishes them luck in the further development of the subject. The text also benefits, as James and Kerber mention in their preface, from conversations with Robinson during extensive visits by the authors to the University of Toronto. The text has yet another brief but interesting foreword by P. M. Cohn.

Chapter 1 of the James-Kerber text is devoted to a preliminary exposition of the point of view which they use to present later material. A reader familiar with some basic representation-theory of finite groups (e.g., induced representations and Mackey's intertwining theorem) should have no excessive difficulty in following this material; a reader working in some other field than group-theory, but wishing to learn about representations of S_n, might do well to study [**26**], and perhaps look at [**2** and **29**], before tackling the present text. The text gets down to business in Chapter 2, which contains a wealth of classical results, many presented in a novel manner. Chapter 3 deals with a selected portion of Young's work; the treatment here does not differ significantly from that in Rutherford [**43**] and Boerner [**2**].

Reasons of space forbid a description of the many topics covered in Chapter 2 alone; however, the content of §2.8, the celebrated Littlewood-Richardson Rule, demands some discussion, if only because of its remarkable history.

Like many results in the theory, the Littlewood-Richardson Rule expresses the dimension of a space of interest in representation-theory, in terms of the number of ways of solving some combinatorial problem involving Young tableaux. It is perhaps best understood in the context of representations of the general linear group:

Let E be a finite-dimensional vector-space over a field of characteristic 0, and let $\bigwedge^\alpha E$, $\bigwedge^\beta E$ denote representation-spaces for the irreducible representations of $GL(E)$ associated with partitions α and β. It is natural to ask how the tensor-product $\bigwedge^\alpha E \otimes \bigwedge^\beta E$ of these two irreducible representations decomposes as a direct sum of irreducible $GL(E)$-modules. The Littlewood-Richardson Rule answers this question, at least in the weak sense of giving the multiplicity with which any $\bigwedge^\gamma E$ occurs in this decomposition: this number turns out to be the number of ways to construct the Young frame for γ out of those for α and β, in accordance with some peculiar requirements.

Littlewood and Richardson [**30**] observed this pattern empirically in 1934, but were unable to prove it. Robinson [**40**] in 1938 showed that the proof may be reduced to the construction of a bijection between two very complicated finite sets, and gave a remarkable rule for constructing such a bijection.

Robinson's construction includes, as a special case, the Robinson-Schensted correspondence (rediscovered by Schensted [**44**] in 1961) which has since been the object of much study. Robinson's argument was elaborated by Littlewood in [**29**, §6.3]. It seems that for a long time the entire body of experts in the field was convinced by these proofs; at any rate it was not until 1976 that McConnell [**32**] pointed out a subtle ambiguity in part of the construction underlying the argument. A large number of complete proofs which appear to avoid this ambiguity has since been published; the earliest (at least in published form) being those of Thomas ([**46**], utilizing ideas of Schützenberger) and James [**22**, §16] while later complete proofs have been published by (at least) Baclawski [**1**], Clausen [**5**], Macdonald [**31**], Zelevinsky [**52**] and the present text. (In addition, three doctoral theses contain new proofs: that of Beetham at Oxford (which pre-dates James' and Thomas' proof), Wagner at Aachen (whose proof is presented in the James-Kerber text) and Weyman at Brandeis). The bulk of these proofs utilize, either the Robinson-Schensted correspondence, or (as in the James-Kerber text) Robinson's construction (related to Young, QSA VIII, §III) of characters associated to skew partitions. Thus, the Littlewood-Richardson Rule seems finally to have been firmly established, by methods based on ideas of Robinson.

How was it possible for an incorrect proof of such a central result in the theory of S_n to have been accepted for close to forty years? The level of rigor customary among mathematicians when a combinatorial argument is required, is (probably quite rightly) of the nonpedantic hand-waving kind; perhaps one lesson to be drawn is that a higher degree of care will be needed in dealing with such combinatorial complexities as occur in the present level of development of Young's approach.

These observations should not be taken as depreciating the intuitions, used to obtain the brilliant results of the earlier period in the subject. Indeed, these earlier feats of pattern-recognition may be better appreciated by turning, as we next shall, to some of the open questions in the field.

The James-Kerber text contains tables (constructed by A. Golembiowski) for the direct-sum decomposition into irreducibles of the tensor-product of two ordinary irreducible representations of S_n. Surely these also must exhibit some pattern, as in the case of $GL(n)$? Perhaps so; but if there is some underlying pattern, we may require some future Littlewood, Richardson and Robinson to bring order out of the chaos of these tables. (This problem is equivalent to that of expressing $S^n(E_1 \otimes E_2 \otimes E_3)$ as a direct sum of terms $\bigwedge^{\alpha_1} E \otimes \bigwedge^{\alpha_2} E \otimes \bigwedge^{\alpha_3} E$, analogously to the well-known direct-sum decomposition

$$S^n(E \otimes F) = \bigoplus_{\alpha \vdash n} \bigwedge{}^{\alpha} E \otimes \bigwedge{}^{\alpha} F;$$

the reviewer is indebted to R. Stanley for this observation.)

The problem just mentioned (cf. discussion in §2.9 of the text) is one of three major open questions in the theory of S_n, the other two being the plethysm problem and the decomposition matrix problem. The curiously named plethysm problem asks for the direct-sum decomposition of $\bigwedge^{\alpha}(\bigwedge^{\beta} E)$ into irreducible $GL(E)$-modules (in characteristic 0, α and β being any partitions). (It has an equivalent formulation in terms of S_n-modules and wreath-products, given in §5.4 of the James-Kerber text; the above formulation, in terms of composites of shape-functors $\wedge^{\alpha}$, is closer to Littlewood's original version [**27**].) Only a few isolated results on this problem are known, e.g., Thrall's [**47**] beautiful decomposition of $S^n(S^2E)$ as the direct sum of all $\wedge^{a_1,a_1,a_2,a_2,\ldots,a_s,a_s}$ with $a_1,\ldots,a_s$ a partition of n (which forms the starting-point of the statistical investigations of A. D. James referred to earlier) (cf. also the tables of plethysms in [**10**]).

The special case of the plethysm problem which asks for the direct-sum decomposition of $S^m(S^nE)$ is related to (still open) questions of nineteenth-century invariant-theory (cf. [**15**, **28**]).

The decomposition matrix problem will be discussed below; it resembles the two preceding problems in asking for the fundamental laws underlying a body of 'experimental data'. One invaluable feature of the James-Kerber text, is that a vast body of 'experimental data' on S_n is assembled, in Appendix I, in the form of 110 pages of tables.

Chapter 4 of the James-Kerber text discusses the representation-theory of wreath-products, and Chapter 5 presents some applications of this theory; these two chapters essentially cover the material in Kerber's earlier text [**23**]. Among the most interesting applications discussed in Chapter 5 may be mentioned: the Redfield-Pólya enumeration theory, results on the plethysm problem and some results of Frobenius on the character-tables of the Mathieu groups. (All these results had earlier been obtained by methods not explicitly involving wreath-products, but it is illuminating to see them presented in this new context.) Chapter 5 also presents some more novel applications of wreath-products, involving operations called 'symmetrization' and 'permutrization' on representations of finite groups.

With the development of modular representation theory in the late 1930s, a new theme arose in the representation-theory of S_n: the problem of explicitly computing for S_n the various general concepts (p-blocks, decomposition matrices, etc.) of the modular theory. Chapter 6 of the text deals with this material.

A beautiful explicit description of the p-blocks for S_n, in a form involving Young frames, was conjectured by Nakayama [**34**] in 1940, and proved by Brauer and Robinson in 1947 [**3** and **41**]; a modern proof is given in §6.1 of the present text.

By contrast, the evaluation of the decomposition matrices for S_n remains a major open question, despite many ingenious attacks on this problem. The pioneeering work of Robinson on this question is explained in his text [**42**]; like later work on the problem by Osima, Farahat, Kerber et al., it deals only with characters and/or isomorphism-classes of S_n-modules. The introduction of Specht modules into the question by Peel (in his thesis [**35**]; cf. [**38**]) led to an important break-through in the study of decomposition matrices of S_n (cf. [**11**, **12**, **24**, **36**, **37**]), and this new method has been extensively developed by both James and Kerber.

§6.3 of the James-Kerber text is a lucid exposition of the current state of the decomposition matrix problem for S_n. Much of this material is the work of the authors, and this section is, in a sense, the high point of the text.

The concluding two sections of the text are devoted to the Young approach (via *modules* rather than *characters*) to the representation-theory of S_n (Chapter 7) and $GL(n)$ (Chapter 8). The approach in both chapters is "*characteristic-free*", (a current buzz-word in the literature) i.e., the constructions and results work over ground-fields of any characteristic.

We have already discussed the importance, for modular representation-theory, of Peel's observation that Specht's construction of explicit representation-modules for S_n is characteristic-free; results in (Peel, [**38**]) imply that Young's earlier construction of such modules and their bases also has this property. James' earlier text [**22**] uses these constructions (in a third, equivalent, incarnation) as a basis for the theory of S_n; in the reviewer's opinion, this approach is more fundamental than that in the present text. Chapter 7 of the present text consists of a small selection of this material from James' earlier text, together with an exposition of unpublished work by Murphy which sheds light on several mysteries at once: Young's orthogonal representation, Nakayama's rule, and the role played by hook-lengths.

Chapter 8 is essentially an exposition of the important paper [**4**] by Carter and Lusztig; the Lie-algebraic material is here replaced by elementary combinatorial constructions, thus making the results accessible to readers unfamiliar with Chevalley bases, Kostant Z-forms and such.

We cannot discuss here more than a small fraction of the many important results and concepts covered in the James-Kerber text. Nevertheless, there are important topics the text omits (notably the theory of Schur functions) and the serious student of S_n will need to consult other current texts (as well as papers); James [**22**], Macdonald [**31**] and Green [**17**] may be particularly recommended for supplementary reading.

James and Kerber have, in the text under review, made accessible to the mathematical community a vast body of information and techniques concerning the representation-theory of S_n and $GL(n)$. Great labor and thought have gone into the systematization of this material. The reviewer believes the subject-matter of their text is at present undergoing an intensive period of development; the James-Kerber text should contribute substantially to the acceleration of this development.

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The logic of quantum mechanics, by Enrico G. Beltrametti and Gianni Cassinelli, Encyclopedia of Mathematics and its Applications, vol. 15, Addison-Wesley, Reading, Mass., 1981, xxvi + 305 pp., $31.50.

By their very nature, scientific theories cannot be proved. No matter how successful a theory has been in explaining the Universe, there always exists the possibility, however remote, that this particular theory is not the only one that can explain the given phenomena. There conceivably could exist another theory that could do just as well—if not better. This possibility is not as remote as it may seem. In the past, very few physical theories have lasted more than a century without being discarded or substantially modified.

Quantum theory was brought about at the turn of the century by the failure of classical physics to explain the results of more accurate experiments which could measure atomic phenomena. The success of the theory was overwhelming, and currently its acceptance among scientists is unquestioned. In the beginning the theory consisted of statements concerning physical quantities, but later writers attempted to axiomatize it and divorce it from concepts of