The title of the book is taken from the 1936 paper [**4**] by Birkhoff and von Neumann, which gave the impetus for much of the research in quantum mechanics. The word "logic" in the title refers to the mathematical foundations of quantum mechanics and not to quantum logic which is mentioned only briefly in the book.

A person unfamiliar with quantum theory will have difficulty reading the book. The authors might have pleased more readers by restructuring the book and including more background material. But the majority of readers will consider it a worthwhile addition to the literature.

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Representations of real reductive Lie groups, by David A. Vogan, Jr., Progress in Mathematics, vol. 15, Birkhäuser, Boston, Basel, Stuttgart, 1981, xvii + 754 pp., $35.00.

1. Introductory. The representation theory of Lie groups is a vast and imposing edifice. Its foundations were laid by E. Cartan in 1913. He gave a classification of the finite-dimensional irreducible representations of a complex semisimple Lie algebra $\mathfrak{g}$ [**2**]. This is the "infinitesimal" version of the classification of finite-dimensional irreducible representations of a semisimple Lie group G. It was realized by H. Weyl in the twenties [**9**] that if G is compact (and connected and simply connected in the topological sense) any continuous irreducible finite-dimensional complex representation of G can be obtained by "integration" from a similar representation of the complexification $\mathfrak{g}$ of the Lie algebra of G. He also showed that any representation of G is equivalent to a unitary one.

Around the same time Peter and Weyl [**6**] showed that these irreducible unitary representations are fundamental objects for noncommutative Fourier analysis on the compact Lie group G. The representation theory of Lie groups is thus tied up with Fourier analysis.

Any attempt at a straightforward generalization of these elegant results to the case of noncompact Lie groups breaks down. To develop Fourier analysis on noncompact Lie groups one needs infinite-dimensional representations of a Lie group G, more precisely continuous representations π of G by bounded operators in a Hilbert space H. Such a representation π is irreducible if no closed nontrivial subspace of H is invariant under all $\pi(x)$ $(x \in G)$.

A host of problems arises in the study of these infinite-dimensional representations. The pioneering work is mainly due to Harish-Chandra, contained in many papers (over a period of more than 30 years, since the late forties; I have refrained from giving references). He also established the fundamental results of Fourier analysis. For this analytic work he needed a great deal of profound information about infinite-dimensional representations. He also constructed and studied concrete infinite-dimensional representations.

In more recent years much work has been done on the algebraization of the theory of infinite-dimensional representations of a semisimple Lie group, which originally had rather an analytic flavour. Basic for this is a result of Harish-Chandra which states that many irreducible representations π have an infinitesimal substratum by which they are determined (see below). The algebraic description of irreducible infinite-dimensional representations is the main theme of Vogan's book. He has made significant contributions to this subject.

After these preliminaries about the setting of the book, I shall try to be a bit more precise, and describe the algebraic objects.

2. Harish-Chandra modules. First I have to be more specific about the groups. Vogan's book deals with "real reductive linear groups". The precise definition is—unavoidably—subtle. For the purpose of this review, the following approximation of the definition is adequate. We consider a Lie group G with the following properties: (a) G is a closed subgroup of some special linear group $SL_n(\mathbf{R})$ with $n > 1$; G has finitely many components, (b) Let θ be the automorphism $x \mapsto {}^t x^{-1}$ of $SL_n(\mathbf{R})$ (^{t} denoting transpose), we assume that G is θ-stable. If K is the fixed point set of θ in G then K is a closed subgroup of G, which is maximal compact. The automorphism of G induced by θ (also denoted θ) is a "Cartan involution" of G. Let $\mathfrak{g}_0$ be the Lie algebra of G, it is a subalgebra of the Lie algebra $\mathfrak{sl}_n(\mathbf{R})$ and let $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C}$ be the complexification. Similarly, let $\mathfrak{k}$ be the complexification of the Lie algebra $\mathfrak{k}_0$ of K; it is a subalgebra of $\mathfrak{g}$. The group G acts naturally on $\mathfrak{g}$, via the adjoint action $\mathrm{Ad}(x)X = xXx^{-1}$, if $x \in G$, $X \in \mathfrak{g}$, similarly for K and $\mathfrak{k}$. Notice that $G = GL_{n-1}(\mathbf{R})$ can be imbedded in $SL_n(\mathbf{R})$, so as to satisfy our requirements. If G is connected, it is a real reductive group in the sense of the definition given on p. 1 of Vogan's book. But if G is not connected, additional technical assumptions on G have to be made. In developing the theory one is forced to work with nonconnected groups G, and also with nonconnected maximal compact subgroups K (the reader will observe that in the example $G = GL_{n-1}(\mathbf{R})$ both G and K are disconnected).

The algebraic substratum of a Hilbert space representation of G is the notion of $(\mathfrak{g}, K)$-*module*. A $(\mathfrak{g}, K)$-module V is a complex vector space, together with a representation of $\mathfrak{g}$ and a (continuous) representation of the compact Lie group K, both representations being denoted by π, such that the following hold:

(a) for each $v \in V$, the vectors $\pi(k)v$ $(k \in K)$ span a finite-dimensional subspace;

(b) the differential of the representation π of K is the restriction to $\mathfrak{k}_0$ of the representation π of $\mathfrak{g}_0$;

(c) if $X \in \mathfrak{g}$, $k \in K$ then $\pi(\mathrm{ad}(k)X) = \pi(k)\pi(X)\pi(k)^{-1}$.

We shall say that the $(\mathfrak{g}, K)$-module V is a *Harish-Chandra module* if V, viewed as a K-module, decomposes into a direct sum of finite-dimensional irreducible subspaces in each of which K acts via a continuous representation, each irreducible representation of K occuring with finite multiplicity. This is not exactly the definition of Harish-Chandra modules of the book, but it coincides with it if V is an irreducible $(\mathfrak{g}, K)$-module. By one of Harish-Chandra's theorems, an irreducible $(\mathfrak{g}, K)$-module is a Harish-Chandra module and can be "integrated" to a Hilbert space representation of G. A basic problem, which is one of the main subjects of this book, is the description and the construction of all irreducible Harish-Chandra modules. This is attacked in the book by first constructing an easier family of Harish-Chandra modules, such that any irreducible one is a submodule of one in the family.

To give an idea of the kind of problems encountered in this attack I shall first describe some of the results of the theory of Verma modules. There one deals with similar questions, but the technical complications are less.

3. Verma modules. In this section the notation will be changed. Now $\mathfrak{g} \subset \mathfrak{sl}_n(\mathbf{C})$ denotes a complex semisimple Lie algebra. Consider the group of all $x \in SL_n(\mathbf{C})$ with $x\mathfrak{g}x^{-1} = \mathfrak{g}$ and let G be its identity component (i.e. the topological connected component containing the identity). Then G is a complex connected semisimple Lie group, with Lie algebra $\mathfrak{g}$. We fix a Borel subgroup B of G and a Cartan subgroup $T \subset B$. If we view G as acting linearly in $\mathbf{C}^n$, then B (resp. T) is a closed connected subgroup of G which is in upper triangular (resp. diagonal) form with respect to some basis of $\mathbf{C}^n$, and which is maximal for these properties. The pair (T, B) is unique up to conjugacy by an element of G. Denote by $\mathfrak{b}$ (resp. $\mathfrak{t}$) the Lie algebra of B (resp. T). They are a Borel subalgebra (resp. Cartan subalgebra) of $\mathfrak{g}$. Notice that $\mathfrak{t}$ is abelian. We also need the universal enveloping algebras of these Lie algebras. For any complex Lie algebra $\mathfrak{g}$, the universal enveloping algebra is a "largest" associative algebra $U(\mathfrak{g})$ with $\mathfrak{g}$ as a subspace and such that the Lie product $[X, Y]$ $(X, Y \in \mathfrak{g})$ equals the commutator $XY - YX$ in $U(\mathfrak{g})$. The algebra $U(\mathfrak{g})$ is unique up to isomorphism. Any $\mathfrak{g}$-module is canonically a $U(\mathfrak{g})$-module, and vice versa.

Verma modules are $\mathfrak{g}$-modules which are constructed by induction from $\mathfrak{b}$. The precise definition is as follows. Let $\mathfrak{t}^*$ be the linear dual of $\mathfrak{t}$. Any $\lambda \in \mathfrak{t}^*$ defines a 1-dimensional representation of $\mathfrak{t}$ and hence of $\mathfrak{b}$ (since $\mathfrak{t}$ is a quotient of $\mathfrak{b}$). Let $\mathbf{C}_\lambda$ be the corresponding 1-dimensional $U(\mathfrak{b})$-module. We put $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_\lambda$ (this makes sense as $U(\mathfrak{b})$ is a subalgebra of $U(\mathfrak{g})$). The $U(\mathfrak{g})$-module M_λ is the *Verma module* defined by λ. These are discussed at length in [**3**]. We mention a few properties.

(a) M_λ has a finite composition series, (b) M_λ has a unique irreducible quotient $L_\lambda \neq 0$. In particular, we have thus a construction of (in general infinite-dimensional) irreducible representations of $\mathfrak{g}$.

The following problem now arises. What are the composition factors of M_λ? In particular, one can ask for which λ we have that $M_\lambda = L_\lambda$ is irreducible. Here the Weyl group W of (G, T) (or $(\mathfrak{g}, \mathfrak{t})$) enters the picture: if N is the normalizer of T in G then $W = N/T$. It is a finite group, acting linearly and faithfully in $\mathfrak{t}^*$ and it is generated by reflections. More precisely, there exist finite subsets $R(R^\vee)$ of $\mathfrak{t}^*$ (resp. $\mathfrak{t}$) and a bijection $\alpha \mapsto \alpha^\vee$ of R onto $R^\vee$ such that the linear maps s_α of $\mathfrak{t}^*$ defined by $s_\alpha x = x - \alpha^\vee(x)\alpha$ are reflections and generate W (viewed as group of linear transformations of $\mathfrak{t}^*$). The elements of R are the *roots* of (G, T).

We then have an *irreducibility criterion*: (c) $M_\lambda = L_\lambda$ if and only if $\alpha^\vee(\lambda + \rho) \leqslant 0$ for α through a certain subset of roots, with ρ as below.

We also need the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. This is a commutative algebra. One shows that there exists a homomorphism χ_λ: $Z(\mathfrak{g}) \mapsto \mathbf{C}$, the *infinitesimal character* of M_λ, such that $z \cdot m = \chi_\lambda(z)m$, for $z \in Z(\mathfrak{g})$, $m \in M_\lambda$. By a theorem of Harish-Chandra, there is $\rho \in \mathfrak{t}^*$ such that $\chi_\lambda = \chi_\mu$ if and only if there is $w \in W$ with $w(\lambda + \rho) - \rho = \mu$. To simplify matters, I shall restrict myself to the case that the character of M coincides with that of the trivial representation (in general, one has similar results, although they may not all have been published yet). In this case we have $\lambda = -w\rho - \rho$, for some $w \in W$. Put $M_w = M_{-w\rho-\rho}$, $L_w = L_{-w\rho-\rho}$. Then: (d) the composition factors of M_w occur among the L_x. In fact, there is an ordering of W such that if L_x is a composition factor of M_w we have $x \leqslant w$.

The ordering of W, the Bruhat order can be described algebraically. However, there is a geometric description which is relevant here. The quotient $X = G/B$ is a projective variety (the flag variety of G). By Bruhat's lemma we have a double coset decomposition $G = \amalg_{w \in W} BwB$, whence a decomposition $X = \amalg_{w \in W} X_w$. Here the Bruhat cell $X_w = BwB/B$ is a locally closed piece of the projective variety X, isomorphic to an affine space $\mathbf{C}^{l(w)}$. The closure $\overline{X}_w$ is called a *Schubert variety*. Now the geometric description of the Bruhat order is thus: we have $x \leqslant w$ if and only if $X_x \subset \overline{X}_w$.

In a suitable Grothendieck group we have a decomposition $[M_w] = \sum_{x \in w} m_{x,w}[L_x]$, with multiplicities $m_{x,w}$. A major recent result is their explicit description. This was conjectured by Kazhdan and Lusztig [**4**], and proved by Beilinson-Bernstein and Brylinski and Kashiwara [**1**]. I shall not go into details here (referring to [**7**] for a less superficial review), and mention only that the $m_{x,w}$ can be described geometrically in terms of a certain numerical measure of the singularity of the variety $\overline{X}_w$ along X_x (in particular, if all points of X_x are smooth in $\overline{X}_w$ then $m_{x,w} = 1$, assuming $x \leqslant w$). The passage from Verma modules to the geometry of X is effected by means of complex analysis (theory of holonomic systems of differential equations on X), and measuring the singularity of X_x along $\overline{X}_w$ is done via the recently developed "intersection cohomology" of singular varieties (due to Goresky-MacPherson and Deligne). From this brief description the reader will have gathered that the problem of describing multiplicities of composition factors of Verma modules is a difficult one, tied up with subtle geometric questions.

4. About Vogan's book. We now return to the situation described in §2, and we use the notations of that section. The purpose of Vogan's book is to give a survey of recent results on Harish-Chandra modules which are analogous to the results on Verma modules just described. In particular, it contains:

(a) a construction of suitable modules, the *standard modules*, from which a description of the irreducible Harish-Chandra modules can be derived. This is mainly done in Chapters 5 and 6 (pp. 217–428);

(b) a discussion of irreducibility of standard representations (Chapters 7–8, pp. 429–619);

(c) a discussion of a conjectural algorithm for computing multiplicities of composition factors of standard modules ("Kazhdan-Lusztig conjecture", Chapter 9, pp. 624–741).

The results of the first 200 pages of the book are mostly preparatory. Below I shall try to give an idea of what is done in the book about these three themes (a), (b), (c).

Recall the situation of §3. The irreducible module L_λ is a quotient of the Verma module M_λ, and M_λ is constructed by algebraic induction from a 1-dimensional representation of the Borel subalgebra $\mathfrak{b}$.

The standard modules are such that any irreducible Harish-Chandra module is a submodule of one of them, and they are also constructed by an induction procedure, which is vastly more complicated than the simple procedure in the Verma module case.

In the set-up used by Vogan (pp. 392–393) it does not suffice to induce from Borel algebras of $\mathfrak{g}$; he needs proper parabolic subalgebras, i.e. subalgebras containing a Borel subalgebra. The parabolic subalgebras $\mathfrak{p}$ needed here have various special properties. We mention the following ones: $\mathfrak{p}$ is stable under the Cartan involution θ (see §2) and the normalizer L of the complex Lie algebra $\mathfrak{p}$ in the real group G is a real reductive linear group, with $L \cap K$ as a maximal compact subgroup. Let $\mathfrak{l}$ be the complexification of the Lie algebra of L. The induction procedure goes from $(\mathfrak{l}, L \cap K)$-modules. The $(\mathfrak{l}, L \cap K)$-modules to be induced are the Harish-Chandra modules defined by certain "principal series" representations of L. These principal series representations are well-known infinite-dimensional representations. The author gives a brief discussion of their basic properties in Chapter 4, using the analytic approach.

The induction procedure used by Vogan is quite subtle. It uses a construction due to Zuckerman, which exploits homological algebra. Let A be a closed subgroup of K and define $(\mathfrak{g}, A)$-modules as in §2. Let $\mathfrak{M}(\mathfrak{g}, A)$ be the category of $(\mathfrak{g}, A)$-modules. One defines (pp. 327–328) first an induction functor Γ from $\mathfrak{M}(\mathfrak{g}, A)$ to $\mathfrak{M}(\mathfrak{g}, K)$, which is left exact. The definition is similar to one for induction of representations of finite groups, but because of infinite dimensionality and nonconnectedness of A and K some care is required. In contrast to the case of induction for ordinary representations of finite groups, the function Γ is not exact, and so has higher derived functions Γ^i. They can be viewed as generalized induction functors. Standard properties of induction become messier for generalized induction, for example Frobenius duality is expressed now by a spectral sequence.

With the above notations, the standard modules are defined using $A = L \cap K$ and one of the corresponding functions Γ^S.

In order to apply this construction usefully, many technical details have to be dealt with. For example, one needs information about the vanishing of $\Gamma^i(M)$ for suitable i and M. All this is done in Chapters 5 and 6 of the book (with some preparatory material in the previous chapters).

Chapter 6 also contains results about the classification of irreducible Harish-Chandra modules (pp. 406–407), and gives the connection with the (unpublished) classification of Langlands, which is more analytic in spirit. It should be pointed out that Vogan's construction and classification are essentially algebraic.

Mention should be made of a basic problem which, as far as I know, has not yet been solved. This is the question of which Harish-Chandra modules come from *unitary* representations.

Let us now turn to theme (b), the analysis of reducibility of standard representations. As in the case of Verma modules, a standard module M has an infinitesimal character ξ. This is now an element of the dual $\mathfrak{t}^*$ of a suitable complex Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, which is contained in $\mathfrak{l}$ and is θ-stable. Then θ acts on the Weyl group W of $(\mathfrak{g}, \mathfrak{t})$ and on the root system R. These actions will enter the picture. Although the root system R is the same (up to isomorphism) for all Cartan subalgebras $\mathfrak{t}$ of the Lie algebra $\mathfrak{g}$, the θ-action depends on $\mathfrak{l}$. An important part of the analysis of reducibility of standard representation is the study of tensor products with finite-dimensional representations. This procedure is also useful in the theory of Verma modules (see **[3]**). The method ("the translation principle") is discussed in Chapter 7 of Vogan's book and it is used in Chapter 8 to obtain results about irreducibility of standard representations. In the discussion a subgroup $W(\xi)$ of W (ξ an infinitesimal character) plays a rôle, in particular an action of $W(\xi)$ on suitable virtual $(\mathfrak{g}, K)$-modules. Here the θ-action on W comes also into play (the way a reflection s_α acts depends on the way θ acts on α).

Finally the last chapter (Chapter 9) of the book which deals with theme (c). It contains a description of an algorithm to compute multiplicities, assuming a conjecture made in Chapter 7 (about complete reducibility of a certain $(\mathfrak{g}, K)$-module). Chapter 9 also contains a description of *blocks* of irreducible Harish-Chandra modules (as in the modular representation theory of finite groups, blocks are equivalence classes of irreducibles for the equivalence relation generated by: $X \sim Y$ if X and Y are composition factors of the same indecomposable). The algorithm of Chapter 9 is called the "Kazhdan-Lusztig conjecture" because it leads to something similar to the multiplicity formula for Verma modules which I mentioned in §3. However, there things were formulated in more geometric terms.

More recently (but this is not to be found in the book) Vogan has succeeded in proving the conjectured algorithm of Chapter 9, for modules whose infinitesimal character is the same as that of a finite-dimensional representation using a geometric approach, as in the case of Verma modules (see **[8]** and also his work with Lusztig **[5]**). The flag variety X introduced in §3 (which can be viewed as the variety of all Borel subalgebras of $\mathfrak{g}$) is again used. Now let $K_{\mathbf{C}}$ be a complexification of the group K. Then $K_{\mathbf{C}}$ acts on X and has finitely many orbits. The geometry of the closures of these orbits comes into play here (as does the geometry of Schubert varieties in the case of Verma modules). Besides, certain locally constant sheaves on these orbits enter the picture (they parametrize irreducible representations). I shall content myself with these very brief indications.

There remain a few final remarks about Vogan's book. It contains much more than I have been able to mention in this review. Although much of the book is oriented towards recent research, some parts of the book are more expository in character, and they are quite good. I think that Chapter 0 gives a good introduction to the basic facts about representations. Chapter 1 discusses the case of $SL_2(\mathbf{R})$ and can be recommended as a very readable description of the infinite-dimensional representations of $SL_2(\mathbf{R})$. It needs few prerequisites. This cannot be said of the book as a whole, though. The subject matter of the book is difficult, and has many ramifications. This imposes high requirements on a prospective reader. To start with, he needs a thorough familiarity with complex Lie algebras and their representations.

The author makes an effort to present things clearly and efficiently, and usually succeeds in achieving this.

It is to be expected that future developments of the theory expounded in Vogan's book will lead to improvements and simplifications. I hope that the book will stimulate readers to find such improvements. Their efforts will be well spent.

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An introduction to homological algebra, by Joseph J. Rotman,[1] Academic Press, New York, 1979, xi + 376 pp., $26.50.

Homological algebra was invented by Henri Cartan and Samuel Eilenberg after World War II. It is essentially a technique borrowed from topology and

[1]The author writes that a complete list of errata for the first printing is available from the Educational Department of Academic Press, New York. All these errors have been corrected in the second printing.