BOOK REVIEWS

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Quantum fields in curved space, by N. D. Birrell and P. C. W. Davies, Cambridge Monographs on Mathematical Physics, vol. 7, Cambridge Univ. Press, New York, 1982, ix + 340 pp., $49.50. ISBN 0-5212-3385-2

Every great physical theory is characterized by one or several universal constants: special relativity gave us c, the speed of light, as a fundamental invariant, quantum mechanics go into effect when actions are comparable to Planck's constant $\hbar$, and the gravitational constant G enters Newtonian mechanics in the law of gravitational attraction as well as in the general theory of relativity as a coefficient to the source of curvature of space-time. The general theory of relativity is actually even more remarkable in that it combines G and c into the formula (Einstein's equations) thereby in principle predicting the behaviour of a system of planets, lightwaves and galaxies. What we don't have is a physical theory which unifies G, c and $\hbar$ into one physical law, i.e. we don't have a quantum theory of gravity.

As is well known, there is a $(c, \hbar)$-theory, namely relativistic quantum field theory with 11-digit predictive power in the case of interactions between photons and electrons. Albeit elusive as a mathematically well defined theory, certain ad-hoc renormalization and perturbation techniques do work in these special cases.

From a physical point of view, quantum gravity, i.e. some $(G, c, \hbar)$-theory, must go into effect at high densities of light and matter with the intuitive effect of dividing the space-time manifold into discrete little chunks of space and time.

Already Planck (1899) pointed out, that from G, c and $\hbar$ one can derive canonical units for length, time and mass, namely

$$l_P = \left(\hbar G/c^3\right)^{1/2} = 1.616 \times 10^{-33}\,\text{cm},$$
$$t_P = \left(\hbar G/c^5\right)^{1/2} = 5.391 \times 10^{-44}\,\text{sec},$$
$$m_P = (\hbar c/G)^{1/2} = 2.177 \times 10^{-5}\,\text{g}.$$

In the delightful book [**8**] one finds the remark that the Planck length l_P is the general limit on the accuracy of position measurements, simply because for a particle of mass m, we cannot localize it with an error less than the Schwartzschild radius $2Gm/c^2$ (the size of the black hole corresponding to m); at the same time, from the Heisenberg uncertainly principle, this error is always bigger than $\hbar/mc$. Equating

$$2Gm/c^2 = \hbar/mc$$

we get $2m^2 = m_P^2$ and the corresponding limit on the position error

$$\hbar/(\hbar c/2G)^{1/2} = 2^{1/2} l_P.$$

Similarly quantum gravity effects turn up when the time and mass scales of the processes to be studied are near the Planck values (such as in the early history of a big bang).

The book under review represents the efforts of one of the schools within axiomatic quantum field theory over the last decade to create a $(G, c, \hbar)$-theory. A completely viable (and presumably spectacular) theory has yet to be found in the future, and even the present attempts are ill-defined and speculative as seen from a mathematical point of view. But they represent intriguing physical ideas and introduce even more theories such as the incorporation of thermodynamics. Recall here the Planck expression for the energy density distribution as a function of the frequency ω,

$$(1) \qquad f(\omega)\,d\omega = \frac{\hbar\omega^3\,d\omega}{\pi^2 c^3[\exp(\hbar\omega/kT) - 1]}$$

for photons at equilibrium inside a box of absolute temperature T (k is Boltzmann's constant). (1) is called a black-body spectrum, and radiation similar to that escaping from such a box (through a small hole) is called thermal radiation. One of the cornerstones of the theory developed in this book is a computation due to Hawking **[5]** predicting thermal radiation emitted from a quantum black hole. Indeed, it has become a generally believed principle that the introduction of nonzero curvature in space-time results in particle production. It is this mechanism that the authors elucidate involving on one hand the quantum theory of fields, characterized by particle numbers being a nonconstant observable, and on the other differential geometry of curved space-time, in particular the classical Einstein equations.

Rather than going into the author's actual physical arguments and computations, it would perhaps be worthwhile to look briefly at the two basic concepts underlying most of the book: (a) Boguliubov transformations and (b) semiclassical Einstein equations.

(a) Consider the axiomatic (nonconstructive) description of the Klein-Gordon field in Minkowski space-time with the usual linear coordinates $x = (t, \mathbf{x}) = (t, x_1, x_2, x_3)$. We start from the classical wave equation

$$(2) \qquad \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + m^2\right)\psi(t, \mathbf{x}) = 0$$

with the positive frequency modes

$$(3) \qquad u_{\mathbf{k}}(t, \mathbf{x}) = \left(\omega(2\pi)^3\right)^{-1/2}\exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]$$

where $\mathbf{k}\cdot\mathbf{x} = k_1x_1 + k_2x_2 + k_3x_3$ and $\omega = (\mathbf{k}\cdot\mathbf{k} + m^2)^{1/2}$. These form an "orthonormal basis" (you'll get your mathematical feathers a little ruffled in this) for the one-particle Hilbert space H with the complex inner product

$$(4) \qquad (\psi_1, \psi_2) = -i\int \psi_1 \frac{\partial\bar{\psi}_2}{\partial t}$$

(integration over a hyperplane $t=$ constant; this is independent of t due to (2)). H may also, via Fourier transform, be realized as L^2 of one sheet of the mass-hyperboloid $k_0^2 - \mathbf{k}\cdot\mathbf{k} = m^2$ with the Lorentz-invariant measure $\omega^{-1}d^3\mathbf{k}$. We have formally

$$(u_{\mathbf{k}}, u_{\mathbf{k}'}) = \delta(\mathbf{k}-\mathbf{k}')$$

where the right-hand side is the 3-dimensional Dirac function. The quantized field observable φ is a distribution on Minkowski space with values in the space of selfadjoint operators on a Hilbert space K, the state space of the quantized field. φ is the quantum analogue of the real part of the classical field, and its initial conditions are expressed as a pair of canonical commutation relations (generalized Heisenberg relations), formally

$$[\varphi(t,\mathbf{x}), \varphi(t,\mathbf{x}')] = 0,$$
$$\left[\frac{\partial\varphi}{\partial t}(t,\mathbf{x}), \frac{\partial\varphi}{\partial t}(t,\mathbf{x}')\right] = 0,$$
$$\left[\varphi(t,\mathbf{x}), \frac{\partial\varphi}{\partial t}(t,\mathbf{x}')\right] = i\delta(\mathbf{x}-\mathbf{x}'),$$

for all t, $\mathbf{x}$, $\mathbf{x}'$. If we expand

$$\varphi(t,\mathbf{x}) = \int\big(a(\mathbf{k})u_{\mathbf{k}}(t,\mathbf{x}) + a(\mathbf{k})^*\bar{u}_k(t,\mathbf{x})\big)d^3\mathbf{k} \tag{5}$$

we get the commutation relations expressed as

$$\begin{aligned}[a(\mathbf{k}), a(\mathbf{k}')] = [a(\mathbf{k})^*, a(\mathbf{k}')^*] &= 0,\\ [a(\mathbf{k}), a(\mathbf{k}')^*] &= \delta(\mathbf{k}-\mathbf{k}'),\end{aligned} \tag{6}$$

where $a(\mathbf{k})^*$ is the adjoint operator to $a(\mathbf{k})$. On the vacuum state vector $|0\rangle \in K$ we have $a(\mathbf{k})|0\rangle = 0$, and the $a(\mathbf{k})^*$ ("creation operators") generate the "orthonormal basis"

$$\left|\mathbf{k}_1^{n_1}\mathbf{k}_2^{n_2}\cdots\mathbf{k}_j^{n_j}\right\rangle = \left[(n_1)!\cdots(n_j)!\right]^{-1/2}(a(\mathbf{k}_1)^*)^{n_1}\cdots\left(a(\mathbf{k}_j)^*\right)^{n_j}\left|0\right\rangle$$

a so-called state with n_1 particles having momentum $\mathbf{k}_1$, n_2 having $\mathbf{k}_2$, etc. All this can be made rigorous (constructive) [**1**, **11**] using the fact that one may integrate distributions against test functions to produce numbers. In particular, K may be realized [**12**] as the space of holomorphic functions on H, square-integrable with respect to a Gaussian measure, and there is a well-defined Hamiltonian (energy operator) corresponding to the formal (and divergent)

$$E(\varphi) = \frac{1}{2}\int\left(\frac{\partial\varphi}{\partial t}\right)^2 + \left(\frac{\partial\varphi}{\partial x_1}\right)^2 + \left(\frac{\partial\varphi}{\partial x_2}\right)^2 + \left(\frac{\partial\varphi}{\partial x_3}\right)^2 + m^2\varphi^2$$

(integration over $t=$ const). Also there is a so-called Weyl system W consisting of a strongly continuous map $W\colon H \to U(K)$ (the unitary operators on K) satisfying

$$W(z)W(z') = \exp(i\,\mathrm{Im}(z,z')/2)W(z+z') \tag{7}$$

and a vector $|0\rangle \in K$ (the constant function on H) such that for all $S \in U(H)$ there is a $\Sigma \in U(K)$ with $\Sigma|0\rangle = |0\rangle$ and

$$\Sigma W(z)\Sigma^* = W(Sz) \tag{8}$$

for all $z, z' \in H$. (7) represents the relations (6) in integrated form, namely for $z = \psi(t, \mathbf{x})$ we have formally

$$W(z) = \exp\left[i\int\left(\varphi(t,\mathbf{x})f(\mathbf{x}) + \frac{\partial\varphi}{\partial t}(t,\mathbf{x})g(\mathbf{x})\right) d^3\mathbf{x}\right]$$

where (f, g) is the real initial data

$$f(\mathbf{x}) = \tfrac{1}{2}(\psi(0,\mathbf{x}) + \bar{\psi}(0,\mathbf{x})),$$
$$g(\mathbf{x}) = \frac{1}{2}\left(\frac{\partial\psi}{\partial t}(0,\mathbf{x}) + \frac{\partial\bar{\psi}}{\partial t}(0,\mathbf{x})\right).$$

If (f', g') similarly is the initial data for $z' = \psi'(t, \mathbf{x})$ then we have

$$\int (f'(\mathbf{x})g(\mathbf{x}) - f(\mathbf{x})g'(\mathbf{x}))\, d^3\mathbf{x} = \operatorname{Im}(z, z')$$

which identifies the canonical time-invariant symplectic form on the space L of real initial data (the classical phase space) with the imaginary part of the complex inner product on H. Thus in our simple example the field quantization consists in the "algebraic aspect" (6) and (7)—essentially representation theory of infinite-dimensional Heisenberg groups—and in the "probabilistic aspect" of finding $|0\rangle$, see **[13]**. Note that the existence of a complex structure on L is crucial for the existence of $|0\rangle$, here related to the existence of a set of positive frequency modes (3). In general space-times we no longer have a nice Fourier transform and plane waves, so there one must study the geometry of certain vector fields similar to $\partial/\partial t$ **[2, 3]** and Green's functions **[7]**. Another possibility is to explore the geometry of the complexification of the space-time: the modes (3) correspond to holomorphic functions in the tube domain $U(2,2)/U(2) \times U(2)$. Of course, there may be no vacuum, e.g. if $m^2 < 0$ above (tachyons).

There are actually several ways of looking (guessing) at the "quantization" of a general (interacting, nonlinear) classical physical system with infinitely many degrees of freedom, in the example above a scalar Klein-Gordon field. The Hamiltonian approach **[4, 6]** looks for the generator of the time evolution as a selfadjoint operator on a Hilbert space similar to K above, usually by successive approximations ("cutoffs"). For Klein-Gordon the time evolution is simply given in H as multiplication by $\exp(-i\omega t)$ with a natural induced action on the holomorphic functions on H. The functional approach has its roots in the calculus of variations and oscillatory integrals. The object here is to compute the vacuum expectation values $\langle B(\varphi)\rangle = \langle 0 | B(\varphi) | 0\rangle$ of functionals (field observables) $B = B(\psi)$ and the formula is

$$\langle B(\varphi)\rangle = \int B(\psi)\exp(iA(\psi))\, d[\psi] \tag{9}$$

where $d[\psi]$ is some measure on the space of all classical configurations $\psi = \psi(x)$ of the field. Here A is the classical action (for Klein-Gordon

$$A(\psi) = \frac{1}{2}\int\left(\frac{\partial\psi}{\partial t}\right)^2 - \left(\frac{\partial\psi}{\partial x_1}\right)^2 - \left(\frac{\partial\psi}{\partial x_2}\right)^2 - \left(\frac{\partial\psi}{\partial x_3}\right)^2 - m^2\psi$$

over all of space-time). Finally, there is the C^*-algebraic approach **[13]** and related to it the point of view **[9]** (as in (7) and (8)) that field quantization primarily associates unitary operators on the quantum field Hilbert space to symplectic isomorphisms of the classical phase space.

This is exactly where Boguliubov transformations come in, both as an important concept in the rigorous discussion of quantum fields, and as a device for particle production in concrete nonrigorous models of particles in curved space. Returning to fields of the form (5) and (6), suppose there was an alternative set of positive frequency modes (3), say $v_{\mathbf{k}}$ with which to decompose the field

$$\varphi(x) = \int \left(b(\mathbf{k}) v_{\mathbf{k}}(x) + b(\mathbf{k})^* \bar{v}_{\mathbf{k}}(x) \right) d^3\mathbf{k}$$

where x is a variable on the space-time manifold. Correspondingly, there would be a new vacuum $|0\rangle'$ with $b(\mathbf{k})|0\rangle' = 0$ but $a(\mathbf{k})|0\rangle' \neq 0$ in general, so that the $|0\rangle'$ vacuum actually contains particles from the $u_{\mathbf{k}}$ modes. This is the mechanism for the celebrated particle production, including thermal radiation (1) from a black hole. The change of basis from the $u_{\mathbf{k}}$ modes (typically "ingoing" in a scattering situation) to the $v_{\mathbf{k}}$ modes ("outgoing") is called a Boguliubov transformation S. Expressing

$$v_{\mathbf{k}}(x) = \int \left(\alpha(\mathbf{k}, \mathbf{k}') u_{k'}(x) + \beta(\mathbf{k}, \mathbf{k}') \bar{u}_{\mathbf{k}'}(x) \right) d^3\mathbf{k}'$$

combined with the orthogonality relations for the two sets of modes, we get on the diagonal in $H \oplus \overline{H}$ (setting $(\bar{u}, \bar{v}) = -(v, u)$)

$$S = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}^t \qquad \text{(transpose)},$$

with α and β viewed as integral operators. Here

$$\beta\alpha^t - \alpha\beta^t = 0 \qquad \beta\beta^* - \alpha\alpha^* = -1,$$

which means that $SJS^t = J$ where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Therefore, S can be thought of as a symplectic isomorphism of the one-particle Hilbert space H, i.e. $\operatorname{Im}(Sz, Sz') = \operatorname{Im}(z, z')$ $(z, z' \in H)$. Clearly, S preserves the relations (7) in that $W'(z) = W(Sz)$ again satisfies them, but it is only unitarily implementable as in (8) provided $[S, i]$ is a Hilbert-Schmidt operator **[14]**. If S is unitary, there will be no particle production, and while the algebraic aspects of the quantum fields are invariant under Boguliubov transformations, the vacuum and with it the probabilistic aspects (and in particular getting numbers out of the theory) need not be. But as we saw, only special wave equations on special space-time manifolds will allow the splitting of fields into positive and negative frequencies. Also note that such an existence of positive modes analogous to (3) and hence a vacuum is related to the stability properties of the equation **[9]**.

(b) Difficult as it may be mathematically to understand the practice of renormalization in quantum electrodynamics (QED), quantization of gravity presents even bigger problems. For one thing, Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi \frac{G}{c^4} T_{\mu\nu} \tag{10}$$

are nonlinear in the metric tensor $g_{\mu\nu}$, and the coupling constant G is not dimensionless as in QED. A semiclassical theory is therefore invented by keeping the left-hand side of (10) as it is, a function of $g_{\mu\nu}$ and its derivatives, and replacing the right-hand side by its vacuum expectation value as in (9),

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi \frac{G}{c^4} \langle T_{\mu\nu} \rangle . \tag{11}$$

$T_{\mu\nu}$ is the stress-energy-momentum tensor of the quantized matter field φ (quadratic in φ and its derivatives), acting as a source of gravity.

A major part of the book deals with the renormalization of $\langle T_{\mu\nu} \rangle$ which contains divergences like $F(x, x)$ where $F(x, y) = (x - y)^{-a}$, $a > 0$. A first-order quantum gravity is erected by allowing quantum fluctuations $\varepsilon_{\mu\nu}$ in the classical background space-time $g_{\mu\nu}$—after all, gravitational radiation is an observational fact **[15]**—and then treating $\varepsilon_{\mu\nu}$ as part of the right-hand side of (11). This marks the present frontier of the subject, very far indeed from the realm of rigorous mathematics (and experimental verification). But here one finds an intriguing blend of intuitive notions of space, time, particles, energy, temperature, entropy, etc. Recently, however, a different line of attack has been initiated using the gauge theory approach to gravity. Here you try to evaluate functional integrals as in (9) by saddle point methods around instantons (metrics stationary for the action) **[10]**.

This book is not a textbook. Rather, (page vii) "we have attempted to collect and unify the vast number of papers that have contributed to the rapid development of this area" in recent years. It reports many computations of Green's functions and singular integrals for fields in concrete space-times (Robertson-Walker, Rindler) and has good sections on the relevant aspects of differential geometry, such as conformal differential geometry. Here you'll also see (or more often, get a reference) how infinities get absorbed, e.g. by renormalizing physical quantities such as energy, mass, charge, wavefunction, gravitational constant G or cosmological constant Λ. Unfortunately, the book gives almost no references to the purely mathematical literature, which however reflects well the fact that in this area physics and mathematics have grown apart (or one ahead of the other).

One might say that we are witnessing the solution of equations whose left-hand side belongs to the mathematics of the 19th and 20th century (differential geometry of Lorentz manifolds, conformally invariant equations), but whose right-hand side is of the 21st or 22nd century. Hopefully by then some constructive theory will exist for interacting quantum fields in 4 space-time dimensions, including G, c and $\hbar$, a theory which properly dequantizes into present-day classical wave equations. Perhaps, as suggested by Yu. I. Manin **[8]**, we are still living with some WKB-type approximation to a "true" infinite-dimensional complex quantum World?

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Non-relativistic quantum dynamics, by W. O. Amrein, Mathematical Physics Studies, vol. 2, Reidel, Dordrecht, Holland; Boston, U. S. A.; London, England, 1981, viii + 237 pp., paper Dfl. 70.00/US $34.50. ISBN 9-0277-1324-3

The author is one of the foremost expositors (see, for example, the more comprehensive book Amrein, Jauch and Sinha [**1**]) of the scattering theory of Schrödinger operators and the present introductory book is a welcome addition. I use the term introductory here purposely but with two qualifications.

First, as the author states, the book is in the spirit of an essay on the so-called time-dependent methods as viable alternatives to the more ponderous time-independent methods. Moreover, after some perusal, one sees that the main thrust of the book is to present a treatment of some of the very most recent research results in the time-dependent potential scattering theory. These go under the name "geometric" methods. They have emerged only recently in the pioneering work of V. Enss [**2**] (see also Enss [**3, 4**]), and have been