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# A GENERALIZATION OF TWO CLASSICAL CONVERGENCE TESTS FOR FOURIER SERIES, AND SOME NEW BANACH SPACES OF FUNCTIONS

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ABSTRACT. The norms of these spaces fill the gap between the uniform and the variation norms. Their duals are described in terms of generalized variation. One application of these spaces is a new convergence test for Fourier series which includes both the Dirichlet-Jordan and the Dini-Lipschitz tests [1].

**1. The $\kappa$-entropy.** $\kappa(s)$ will always denote a nondecreasing concave function on $[0,1]$ such that $\kappa(0)=0$, $\kappa(1)=1$; this implies that $\kappa(s)$ is continuous except, perhaps, at $s=0$.

DEFINITION. Let $E=\{x_1<x_2<\cdots<x_n\}\subset[a,b]$ be a finite nonempty set. The following quantity will be called the *$\kappa$-entropy* of $E$ (relative to $[a,b]$):

$$\kappa(E)=\kappa(E;[a,b])=\sum_1^{n+1}\kappa((x_j-x_{j-1})/(b-a)), \tag{1}$$

where $x_0=a$, $x_{n+1}=b$. For an arbitrary closed set $F\subset[a,b]$ we set

$$\kappa(F)=\kappa(F;[a,b])=\sup\{\kappa(E)\colon E\subset F \text{ finite}\}. \tag{2}$$

Finally, we set $\kappa(\varnothing)=0$.

The following properties of the $\kappa$-entropy are easily derived.

(i) $F_1\subset F_2$ implies $\kappa(F_1)\le\kappa(F_2)$.

(ii) $\kappa(F_1\cup F_2)\le\kappa(F_1)+\kappa(F_2)$.

(iii) If card $E=n$, then $\kappa(E)\le(n+1)\kappa(1/(n+1))$; the estimate is sharp and attained for $x_1-x_0=x_2-x_1=\cdots=x_{n+1}-x_n$.

**2. Examples of $\kappa$-entropy.**

(a) $\kappa(s)=s$. We have in this case $\kappa(F)=1\,(F\ne\varnothing)$, $\kappa(\varnothing)=0$.

(b) $\kappa(s)=1\,(0<s\le1)$. Here we have

$$\kappa(F)=\operatorname{card}(F\cup\{a,b\})-1 \qquad (F\ne\varnothing).$$

(c) $\kappa(s)=s(1-\log s)$. The corresponding entropy will be denoted by $\kappa_s(F)$ and called the *Shannon entropy* of $F$ (relative to $[a,b]$).

(d) $\kappa(s)=s^\alpha$. Here $\kappa(F)=\kappa_{l,\alpha}(F)$ is the *Lipschitz entropy* $(0<\alpha<1)$.

(e) $\kappa(s)=(1-\frac12\log s)^{-1}$; $\kappa(F)=\kappa_d(F)$ is the *Dini entropy.*

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**3. The $\kappa$-entropy norm.**

DEFINITION. The *$\kappa$-entropy* norm (or simply the $\kappa$-norm) of a real continuous function $x(t)$ on $[a,b]$ is

$$\|x\|_\kappa = \|x\|_C + \int_{-\infty}^{\infty} \kappa(E_y, [a,b])\, dy, \tag{3}$$

where $\|x\|_C = \max\{|x(t)|:\, a \le t \le b\}$ and $E_y = E_y[x] = \{t \in [a,b]:\, x(t) = y\}$ is the level set of $x(t)$.

EXAMPLES. (a) $\kappa(s) = s$. Here we have $\|x\|_\kappa = \|x\|_C + \max x(t) - \min x(t)$; thus $\|x\|_C \le \|x\|_\kappa \le 3\|x\|_C$, so that the $\kappa$-norm in this case is equivalent to the uniform norm.

(b) $\kappa(s) = 1\,(0 < s \le 1)$. We have

$$\|x\| = \|x\|_C + \int_m^M (\text{card } E_y + 1)\, dy = \|x\|_C + M - m + \text{Var } x,$$

where $M = \max x(t)$, $m = \min x(t)$; thus

$$\|x\|_C + \text{Var } x \le \|x\|_\kappa \le 3\|x\|_C + \text{Var } x.$$

(c) The $\kappa$-norm corresponding to the Shannon, Lipschitz and Dini entropies is denoted respectively by $\|\cdot\|_s$, $\|\cdot\|_{l,\alpha}$ and $\|\cdot\|_d$ and called the Shannon-, Lipschitz- and Dini-entropy norm.

In what follows we assume that $\kappa(0^+) = 0$ and $\kappa(s)/s \to \infty\,(s \to 0)$, since otherwise the $\kappa$-norm is equivalent either to the $C$-norm or the $V$-norm.

**4. The spaces $C_\kappa[a,b]$.**

THEOREM 1. *Every $\kappa$-norm is homogeneous and convex: $\|\lambda x\|_\kappa = |\lambda|\,\|x\|_\kappa$, $\|x_1 + x_2\| \le \|x_1\|_\kappa + \|x_2\|_\kappa$. Equipped with a $\kappa$-norm, the linear set of all real continuous functions $x(t)$ on $[a,b]$ such that $\|x\|_\kappa < \infty$ forms a (real) Banach space $C_\kappa[a,b]$; this space is separable: polynomials are dense in $C_\kappa[a,b]$.*

The homogeneity of the $\kappa$-norm follows directly from the definition; however, the proof of the triangle inequality is more difficult.

**5. The $\kappa$-variation.**[2]

DEFINITION. The *$\kappa$-variation* of a real function $\mu(t)$ over $[a,b]$ is

$$\text{Var}_\kappa\, \mu = \text{Sup}\left\{\left(\sum_1^{n+1} |\mu(x_j) - \mu(x_{j-1})|\right)\Big/ \kappa(E; [a,b])\right\}, \tag{4}$$

where the supremum is taken over all finite sets

$$E = \{x_1 < x_2 < \cdots < x_n\} \subset [a,b] \text{ and } x_0 = a,\ x_{n+1} = b.$$

It is easily seen that $\text{Var}_\kappa\, \mu < \infty$ implies the existence of unilateral limit values $\mu(t^+)\,(a \le t < b)$ and $\mu(t^-)\,(a < t \le b)$. Every such function $\mu(t)$ generates a "measure" on the set of all intervals $I \subset [a,b]$, e.g. $\mu([\alpha,\beta]) = \mu(\beta^+) - \mu(\alpha^-)$, $\mu(\alpha,\beta) = \mu(\beta^-) - \mu(\alpha^+)$, and so on. If $\text{Var}_\kappa\, \mu < \infty$, then this measure can be extended to all (relatively) open sets $G \subset [a,b]$ such that

[2]This notion was first introduced in [**2**] for the Shannon variation (see also [**3**]).

$\kappa(\partial G) < \infty$ by the formula $\mu(G) = \sum_j \mu(I_j)$, where $I_j$ are the components of $G$; the series is absolutely convergent. Similarly, for closed sets $F \subset [a,b]$ we define $\mu(F) = \mu([a,b]) - \mu([a,b] \setminus F)$.

The linear set consisting of all $\mu(t)\,(a \leq t \leq b)$ such that $\mathrm{Var}_\kappa \mu < \infty$, provided with the norm $\|\mu\| = \mathrm{Var}_\kappa \mu$, is a Banach space $V_\kappa[a,b]$; for the special cases of the Shannon, Lipschitz or Dini variation this space is denoted respectively by $V_s$, $V_{l,\alpha}$ and $V_d$.

**6. The $\kappa$-integral.**

DEFINITION. Let $x(t) \in C_\kappa[a,b]$ and $\mu(t) \in V_\kappa[a,b]$. The *$\kappa$-integral* of $x$ with respect to $d\mu$ is defined as follows:

$$\int_a^b x(t)\,d\mu(t) = m\mu([a,b]) + \int_m^M \mu(F_y[x])\,dy, \tag{5}$$

where $m = \min x(t)$, $M = \max x(t)$, and $F_y[x] = \{t \in [a,b] \colon x(t) \geq y\}$ are the Lebesgue sets of $x(t)$.

It is easily seen that, by (3) and (4), $\mu(F_y)$ is summable over $(m, M)$; we also deduce

$$\left|\int_a^b x(t)\,d\mu(t)\right| \leq \|x\|_\kappa \,\mathrm{Var}_\kappa\, \mu. \tag{6}$$

If $\mu$ is of bounded (classical) variation, then $\int x\,d\mu$ exists as a Riemann-Stieltjes integral and its value coincides with that of the $\kappa$-integral.

**7. The dual of $C_\kappa$.**

THEOREM 2. *$V_\kappa$ is the dual of $C_\kappa$. This means that every linear functional $F(x)$ in $C_\kappa[a,b]$ has the form of a $\kappa$-integral* (5), *where $\mu$ is uniquely* (*up to a constant*) *determined by $F$. We also have $\frac{1}{3}\mathrm{Var}_\kappa\, \mu \leq \|F\| \leq \mathrm{Var}_\kappa\, \mu$.*

**8. A convergence test for Fourier series.** The Dirichlet-Jordan (D-J) convergence test [**1**] states that the (symmetrical) partial sums $S_n(t;f)$ of the Fourier series of a $2\pi$-periodic function $f(t)$ of bounded variation tend to $\frac{1}{2}[f(t+0) + f(t-0)]$ as $n \to \infty$; if $f(t)$ is also continuous, then $S_n(t) \to f(t)$ uniformly.

The Dini-Lipschitz (D-L) test [**1**] states that $S_n(t;f) \to f(t)$ uniformly if the modulus of continuity $\omega(\delta)$ of $f(t)$ is $o(|\log \delta|^{-1})\,(\delta \to 0)$.

The proof of the Dirichlet-Jordan test is based on the C-V duality. However, if instead of the C-V duality we use the Dini-entropy-norm—Dini-variation duality ($\mathrm{C_d}$-$\mathrm{V_d}$), we obtain a new test that includes both the D-J and the D-L tests.

DEFINITION. A function $\mu(t) \in V_\kappa[a,b]$ is said to be of *vanishing $\kappa$-variation* at $t_0 \in [a,b]$ if $\mathrm{Var}_\kappa\{(\mu(t) - \mu(t_0))\chi_\delta(t)\} \to 0\ (\delta \to 0)$, where $\chi_\delta(t)$ is the characteristic function of $[t_0 - \delta, t_o + \delta]$, and the $\kappa$-variation is taken over $[a,b]$. If this takes place at every point $t_0 \in [a,b]$, then $\mu(t)$ is said to be *of vanishing $\kappa$-variation on* $[a,b]$.

REMARK. For the classical variation, if $f(t)$ is of bounded variation on $[a,b]$ and continuous at $t_0$, then $f(t)$ is of vanishing variation at $t_0$. However, for the $\kappa$-variation this is generally not true.

THEOREM 3. *Let $f(t) \in V_d[0,2\pi]$ be $2\pi$-periodic and normalized so that $f(t) = \frac{1}{2}[f(t+0)+f(t-0)]$. If $\varphi(\tau;t_0) = \frac{1}{2}[f(t_0+\tau)+f(t_0-\tau)]$ is of vanishing Dini variation at $\tau = 0$, then the Fourier series of $f(t)$ converges at $t_0$ to $f(t_0)$. If $f(t)$ is of vanishing Dini-variation on $[0,2\pi]$, then $S_n(t,f) \to f(t)\ (n \to \infty)$ uniformly.*

A SHORT OUTLINE OF THE PROOF. We have

$$S_n(t_0;f) - f(t_0) = \int_0^\pi \mathcal{E}_n(t)\, d[\varphi(t;t_0) - f(t_0)], \tag{7}$$

where

$$\mathcal{E}_n(t) = \int_t^\pi D_n(\tau)\, d\tau \quad (0 \le t < \pi), \qquad D_n(\tau) = \sin\left(n+\frac{1}{2}\right)\tau \Big/ \left(\pi \sin\frac{\tau}{2}\right).$$

A simple computation shows that $\mathcal{E}_n$ satisfies

$$|\mathcal{E}_n(t)| \le \min\{1, 4((2n+1)t)^{-1}\} \qquad (0 \le t \le \pi)$$

and is monotone in each of the intervals $(2k\pi/(2n+1), 2(k+1)\pi/(2n+1))(k = 0,1,\dots,n-1)$ and $(2n\pi/(2n+1),\pi)$; from this we deduce that the Dini-entropy norms $\|\mathcal{E}_n\|_d\ (n = 1,2,\dots)$ are bounded if taken over $[0,\pi]$, and tend to 0 if taken over $[\delta,\pi]\ (\delta > 0)$. Using this, and (7) and (6), we get the required result.

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