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SURGERY AND BORDISM INVARIANTS

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Introduction. The approach used here to relate the two subjects in the title is best explained in terms of three "machines".

Machine (1) is the "L-theory machine ", or "surgery machine"; on being fed a discrete group G and homomorphism $w\colon G \to Z_2$, it produces a spectrum $\mathbb{L}_{\cdot}(G,w)$ whose homotopy groups are the surgery obstruction groups (choose your favourite version),

$$\pi_n(\mathbb{L}_{\cdot}(G,w)) = L_n(G,w) \quad \text{for } n \in \mathbf{Z}.$$

Machine (2) is the "bordism theory machine": on being fed a CW-space B and vector bundle γ on B, it produces a bordism spectrum (or Thom spectrum) $M(B,\gamma)$. The homotopy groups $\pi_n(M(B,\gamma))$ are the bordism groups of closed smooth manifolds N^n equipped with a bundle map from the normal bundle ν_N to γ.

This note will describe a third machine, obtained by welding together the previous two. (The aim is to extend the theory of the "generalized Kervaire invariant": cf. [**1**, **2**].)

Description of Machine (3).

Input. The following input data are required:

— a group G and homomorphism $w\colon G \to Z_2$, as for Machine (1);

— a CW-space B and bundle γ on B, as for Machine (2);

— a principal G-bundle α on B and an identification j of the two double covers of B arising from these data. (They are the orientation cover associated with γ, and the double cover induced from α via w.)

Output. Machine (3) produces a spectrum $\mathbb{L}^{\cdot}(G,w;B,\gamma;\alpha,j)$ (informally: $\mathbb{L}^{\cdot}(B,\gamma)$) and maps of spectra

$$\mathbb{L}_{\cdot}(G,w) \to \mathbb{L}^{\cdot}(B,\gamma) \leftarrow M(B,\gamma).$$

Like Machines (1) and (2), Machine (3) is functorial: Given two input strings $(G,w;B,\gamma;\alpha,j)$ and $(G',w';B',\gamma';\alpha',j')$, and

— a map $f\colon B \to B'$ covered by a bundle map $\gamma \to \gamma'$;

— a homomorphism $h\colon G \to G'$ so that $w' \cdot h = w$;

— an identification of principal G'-bundles on B,

$$h_*(\alpha) \cong f^*(\alpha'),$$

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compatible with the bundle map as regards j and j', Machine (3) yields a commutative diagram

$$\begin{array}{ccccc} \mathbb{L}_{:}(G,w) & \to & \mathbb{L}^{\cdot}(B,\gamma) & \leftarrow & M(B,\gamma) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}_{:}(G',w') & \to & \mathbb{L}^{\cdot}(B',\gamma') & \leftarrow & M(B',\gamma'). \end{array}$$

More surprising is the following property. Write (informally) $\hat{L}^n(B,\gamma)$ for the nth homotopy group of the map of spectra

$$\mathbb{L}_{:}(G,w) \to \mathbb{L}^{\cdot}(G,w;B,\gamma;\alpha,j).$$

THEOREM. *There is a functorial long exact sequence*

$$\cdots \to \hat{Q}^{n+1}(C(\tilde{B})) \to \hat{L}^n(B,\gamma) \to Q^n(C(\tilde{B})) \to \hat{Q}^n(C(\tilde{B})) \to \hat{L}^{n-1}(B,\gamma) \cdots$$
$$(n \in \mathbf{Z}).$$

(*Explanation*: $C(\tilde{B})$ is the cellular chain complex of the total space of α — regarded as a chain complex of left projective modules over the ring with involution $A := \mathbf{Z}[G]$. The chain homotopy invariant functors $Q^n(-)$, $\hat{Q}^n(-)$ are defined on the category of such chain complexes, as follows.

Let C be any chain complex of left projective A-modules; then, using the involution on A, C can be regarded as a chain complex of right A-modules, written C^t. So $C^t \otimes_A C$ is defined, and is a chain complex of $\mathbf{Z}[Z_2]$-modules; the generator $\tau \in Z_2$ acts by switching factors, with the usual sign rules.

Define the $\mathbf{Z}[Z_2]$-module chain complexes W, $\hat{W}$ as follows:

$$W_r := \mathbf{Z}[Z_2] \quad \text{for } r \geq 0, \qquad W_r = 0 \quad \text{for } r < 0,$$
$$d\colon W_r \to W_{r-1}; \quad x \mapsto (1 + (-)^r\tau)\cdot x \quad \text{for } r > 0,$$

and $\hat{W}_r := \mathbf{Z}[Z_2]$ for $r \in \mathbf{Z}$,

$$d\colon \hat{W}_r \to \hat{W}_{r-1}; \quad x \mapsto (1 + (-)^r\tau)\cdot x \quad \text{for } r \in \mathbf{Z}.$$

Now let $Q^n(C)$ and $\hat{Q}^n(C)$ be the nth homology groups of the chain complexes $\mathrm{Hom}_{\mathbf{Z}[Z_2]}(W, C^t \otimes_A C)$ and $\mathrm{Hom}_{\mathbf{Z}[Z_2]}(\hat{W}, C^t \otimes_A C)$, respectively. (See [**4**] for more details.)

Ingredients of the construction. The spectra $\mathbb{L}^{\cdot}(G,w;B,\gamma;\alpha,j)$ are algebraic bordism spectra. Their construction is inspired by the following "dictionary" (A is a ring with involution):

chain complex D of projective A-modules — space;

symmetric algebraic Poincaré complex over A — closed manifold;

chain bundle on D — vector bundle on a space.

Here a *chain bundle* on a chain complex D is a 0-dimensional cycle in

$$\mathrm{Hom}_{\mathbf{Z}[Z_2]}(\hat{W}, (D^{-*})^t \otimes_A D^{-*})$$

(and D^{-*} is the "dual" of D, cf. [**4**]). It represents, but should not be confused with, an element in $\hat{Q}^0(D^{-*})$.

Symmetric algebraic Poincaré complexes (C, φ) (of dimension n) are defined in [**4**] (or [**3**]); I insist that φ be an n-dimensional cycle in $\mathrm{Hom}_{\mathbf{Z}[Z_2]}(W, C^t \otimes_A C)$ (whereas [**4**] only requires a class in $Q^n(C)$). On the other hand, C is allowed to be nontrivial in negative dimensions (less restrictive than [**4**]).

Only the third entry of the dictionary is new, but even so it is strongly suggested by [**4**, Part II, Proposition 9.3].

$\underline{\mathbb{L}}^{\cdot}(G, w; B, \gamma; \alpha, j)$ is obtained in two steps: first, passage from B, γ to a chain complex $C(\tilde{B})$ with chain bundle $c(\gamma)$; second, construction of an algebraic bordism spectrum associated with $C(\tilde{B})$ and $c(\gamma)$, using the dictionary above. So the homotopy groups of $\underline{\mathbb{L}}^{\cdot}(G, w; B, \gamma; \alpha, j)$, informally written $L^n(B, \gamma)$, are the bordism groups of symmetric algebraic Poincaré complexes (C, φ) equipped with a classifying chain map $C \to C(\tilde{B})$ which is covered by a "chain bundle map" from the "normal chain bundle of (C, φ)" to $c(\gamma)$.

REMARKS. (i) If (C, φ) is a symmetric algebraic Poincaré complex, the chain complex C carries a "normal chain bundle" (easy to construct). It is well defined up to an infinity of higher homologies (resembling the normal bundle of a geometric manifold, which is well defined up to an infinity of higher concordances).

(ii) Given a string of data $(G, w; B, \gamma; \alpha, j)$, we are first of all faced with the problem of constructing a chain bundle on $C(\tilde{B})$ (the image $c(\gamma)$ of γ in the chain complex world). The previous remark shows how to proceed if B happens to be a closed manifold, γ its normal bundle; functoriality dictates the rest.

(iii) The theorem is obtained using algebraic surgery in the style of [**4**].

(iv) If B is empty, $\underline{\mathbb{L}}^{\cdot}(G, w; B, \gamma; \alpha, j) \simeq \underline{\mathbb{L}}_{\cdot}(G, w)$ as follows from the theorem.

By-products. (i) The theory also gives a homological description of the homotopy groups of the forgetful map

$$J\colon (\text{quadratic } L\text{-theory}) \to (\text{symmetric } L\text{-theory}).$$

(For a fixed ring with involution A, the homotopy groups of the quadratic L-theory spectrum of A are the Wall groups $L_n(A)$. Those of the symmetric L-theory spectrum are the groups $L^n(A)$ of [**4**] (or [**3**]); their main application is to problems of the following kind. "The product of an n-dimensional surgery problem with an m-dimensional closed manifold is an $(n+m)$-dimensional surgery problem; how are the surgery obstructions of the two surgery problems related?" See [**4**] for details.)

Here is the philosophy behind the homological description: If the notion of chain bundle (on a chain complex of left projective A-modules) is any good, there ought to exist a suitable chain complex D and a *universal chain bundle* μ on D (even though D, μ may not correspond to any geometric reality). The associated algebraic bordism theory should be symmetric L-theory, and the long exact sequence of the theorem above should remain valid (with $\hat{L}^n(B, \gamma)$ replaced by $\pi_n(J)$, and $C(\tilde{B})$ by D). All this is true if carefully interpreted.

(ii) There is a version of the theory where orientations are ignored (because Z_2 is used as coefficient ring instead of $\mathbf{Z}$); a typical "input string" for Machine (3) would then have the form $(G; B, \gamma; \alpha)$.

Investigating this modified Machine (3) with $G = \{1\}$, one finds that it reproduces more or less the "generalized Kervaire invariants" of [**1**, **2**]. More precisely:

Suppose that the $(k+1)$st Wu class of γ is zero; then there is a commutative diagram

$$\begin{array}{ccc} \pi_{2k}(M(B,\gamma)) & \to & Z_8 \\ & & \\ & L^{2k}(B,\gamma) & \end{array}$$

(with $L^{2k}(B,\gamma) = \pi_{2k}(\underline{L}^{\cdot}(\{1\}; B, \gamma; \mathrm{id})))$ in which the horizontal arrow is the invariant of [**2**]. The homomorphism from $L^{2k}(B,\gamma)$ to Z_8 is obtained by imitating [**2**]: the elements of $L^{2k}(B,\gamma)$ are represented by $2k$-dimensional symmetric algebraic Poincaré complexes (C,φ) (over $A = Z_2$) with a certain structure, and the said structure permits to refine the nondegenerate symmetric bilinear form on $H^k(C;Z_2)$ to a *quadratic form* with values in Z_4.

(Choices are necessary to make this work; but [**2**] also uses certain choices, and there is a canonical one-one correspondence between the two kinds of choices.)

To summarize, there is a good case for regarding the homomorphism

$$\pi_{2k}(M(B,\gamma)) \to L^{2k}(B,\gamma)$$

itself as "the" generalized Kervaire invariant: it requires no choices or restrictions of any kind and, more important, it gives very slick product formulae. The computation of the groups $L^{2k}(B,\gamma)$ is easy in the case at hand (use the theorem, and bear in mind that any chain complex over Z_2 is homotopy equivalent to its homology).

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