

BOOK REVIEWS

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Applications of centre manifold theory, by J. Carr, Applied Mathematical Sciences, vol. 35, Springer-Verlag, New York, 1981, 142 pp., \$14.00. ISBN 0-3879-0557-4

Theory and applications of Hopf bifurcation, by B. D. Hassard, N. D. Kazarinoff and Y.-H. Wan, London Mathematical Society Lecture Notes Series, No. 41, Cambridge Univ. Press, Cambridge, England, 1981, vi + 311 pp., \$35.00. ISBN 0-5212-3158-2

Elementary stability and bifurcation theory, by Gerard Iooss and Daniel D. Joseph, Springer-Verlag, New York, 1980, xv + 286 pp., \$22.00. ISBN 0-3879-0526-X

Monsieur Jourdain: Par ma foi, il y a plus de quarante ans
que je dis de la prose, sans que j'en susse rien.

Molière, *Le Bourgeois Gentilhomme*, Act II, Scene 4.

Many of the questions now studied in bifurcation theory can be illustrated by means of the following pair of ordinary differential equations for θ and ϕ :

$$(1) \quad \begin{aligned} 2\ddot{\theta} + \ddot{\phi} \cos(\phi - \theta) - \dot{\phi}^2 \sin(\phi - \theta) + \beta_1 \dot{\theta} + \beta_2(\dot{\theta} - \dot{\phi}) \\ + 2\theta - \phi + \lambda \sin(\phi - \theta) = 0, \\ \ddot{\theta} \cos(\phi - \theta) + \ddot{\phi} + \dot{\theta}^2 \sin(\phi - \theta) + \beta_2(\dot{\phi} - \dot{\theta}) + \phi - \theta = 0. \end{aligned}$$

Here the superposed dot denotes the derivative with respect to t , interpreted as time. $\beta_1, \beta_2, \lambda$ are real parameters with $\beta_1 \geq 0, \beta_2 \geq 0, \lambda > 0$. This system of equations describes the motion of the double pendulum, shown in Figure 2, consisting of two weightless rigid shafts OA and AB of unit length capped by particles of unit mass at A and B under the action of a compressive force of constant magnitude λ acting along the line BA . The rotations of OA from OC and of AB from OA are opposed by torsional springs each with spring constant 1 and by torsional dampers with constants β_1 and β_2 . (The plane of this system is taken to be horizontal so that effects of gravity do not appear. Any problem for a pendulum like that of Figure 2 with equal masses, with shafts of equal lengths, and with springs of equal strengths can be reduced to (1) by a suitable scaling.)

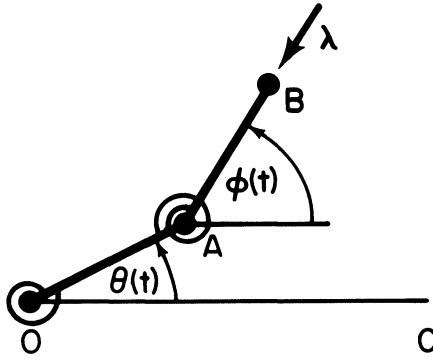


FIGURE 2

It is easy to see that the only equilibrium solution (i.e., solution independent of time) of (1) is the trivial solution $\theta = 0 = \phi$. (If λ were required to act parallel to OC , then for λ sufficiently large there would be nontrivial equilibrium solutions, called *buckled states*, which bifurcate from the trivial branch of solutions. In the early twentieth century, the fact that equilibrium solutions of systems like (1) do not include buckled states was misinterpreted by some to mean that the trivial solution is stable.)

A traditional approach for studying the stability of the trivial solution, hallowed by nearly a century of engineering practice, is to determine whether solutions of the linearization of (1) about the trivial solution remain bounded. The (formal) linearization of (1) in terms of the new variables Θ and Φ is

$$(3) \quad \begin{aligned} 2\ddot{\Theta} + (\beta_1 + \beta_2)\dot{\Theta} + (2 - \lambda)\Theta + \ddot{\Phi} - \beta_2\dot{\Phi} - (1 - \lambda)\Phi &= 0, \\ \ddot{\Theta} - \beta_2\dot{\Theta} - \Theta + \ddot{\Phi} + \beta_2\dot{\Phi} + \Phi &= 0. \end{aligned}$$

The solutions of (3) are found by taking Θ and Φ proportional to $e^{\sigma t}$. The condition that the solution be nontrivial yields the following characteristic equation for σ :

$$(4) \quad \sigma^4 + (\beta_1 + 5\beta_2)\sigma^3 + [\beta_1\beta_2 + 2(3 - \lambda)]\sigma^2 + (\beta_1 + \beta_2)\sigma + 1 = 0.$$

Classical results from the theory of ordinary differential equations yield the following information: If a root σ of (4) has a positive real part, then the trivial solutions of (3) and (1) can be shown to be unstable (in the sense of Lyapunov). If all roots σ of (4) have negative real parts, then the trivial solutions of (3) and (1) can be shown to be asymptotically stable. If all the roots of (4) have nonpositive real parts, with the purely imaginary roots being simple, then the trivial solution of (3) is stable.

In Figure 5 we illustrate the way the roots of (4) move about the complex σ -plane as λ is increased from 0 to ∞ with β_1 and β_2 fixed in the ways indicated. For the undamped case, in which $\beta_1 = 0 = \beta_2$, Figure 5a shows that roots σ of (4) lie on the imaginary axis for $\lambda \in [0, 2]$. The trivial solution of the *linear* equations (3) remains stable until λ reaches the critical value 2 and becomes unstable thereafter. From these facts alone we cannot conclude

anything definitive about the stability of the trivial solution of (1) for $\lambda \in [0, 2]$. By introducing damping about the support O , we remove this ambiguity. From Figure 5b, we conclude that the trivial solutions of (1) and (3) are asymptotically stable for $\lambda < 2$ and are unstable for $\lambda > 2$. The introduction of further damping about the joint A has the striking destabilizing effect shown in Figure 5c. The critical value of λ is reduced to $4/3 + \beta^2/2$.

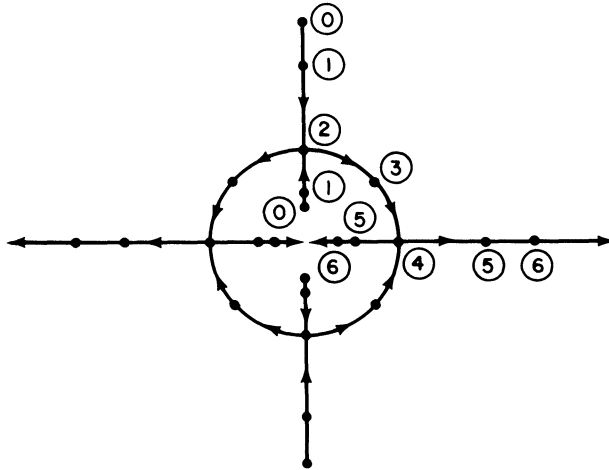


FIGURE 5a. Trajectories of the roots of (4) in the complex σ -plane as λ increases from 0 to ∞ when $\beta_1 = 0 = \beta_2$. The circled numbers associated with the dots are the corresponding values of λ . The circle has radius 1.

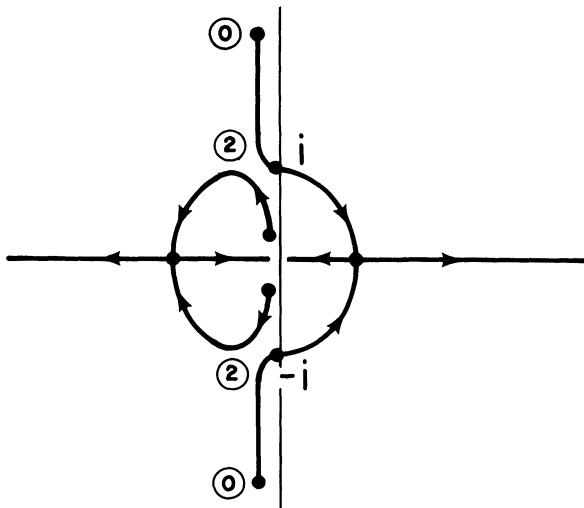


FIGURE 5b. Corresponding trajectories when β_1 is small and positive and $\beta_2 = 0$.

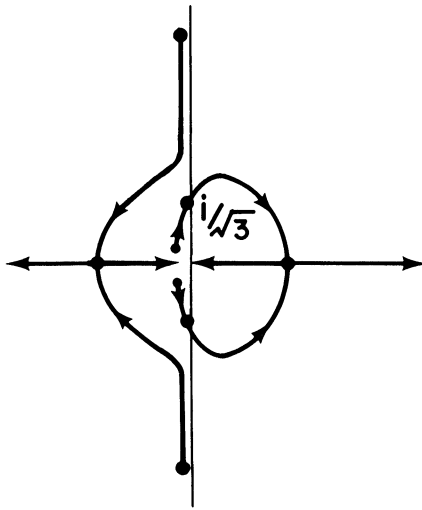


FIGURE 5c. Corresponding trajectories when $\beta_1 = \beta_2 = \beta$ is small and positive. $\lambda = 4/3 + \beta^2/2$ when $\sigma = \pm i/\sqrt{3}$.

The nature of Figures 5b and 5c enables us to deduce further important information about (1). We can parametrize each trajectory by λ until it hits the real axis. Let λ_c denote the critical value of λ . In each case, there is a complex conjugate pair of roots $(\sigma(\lambda), \bar{\sigma}(\lambda))$ with $\sigma(\lambda) = \xi(\lambda) + i\eta(\lambda)$, $\xi(\lambda_c) = 0$, $\xi'(\lambda_c) > 0$, $\eta(\lambda_c) > 0$ and with the real parts of all other roots at λ_c being negative. These observations form the critical hypotheses of the Hopf Bifurcation Theorem, from which we conclude that at $\lambda = \lambda_c$ a branch of periodic solutions of (1) bifurcates from the trivial branch of solutions. A study of the perturbation equations, which we do not carry out, then yields information on the disposition of the bifurcating branch in the space of $(\lambda, (\theta, \phi))$ and on the asymptotic stability of the solutions on the branch. Such local results can be supplemented by global results on Hopf bifurcation due to Alexander and Yorke (1978) and Ize (1976). In this review we limit our attention to the local theory.

A comparison of the three trajectories of Figure 5 suggests that the right way to study the problem is to examine the behavior of solutions as (β_1, β_2) ranges over the closed first quadrant. (The careful investigator would also examine effects of other parameters that we have set equal to 1.) Analyses of this sort have been performed by Takens (1974a, b) and for a problem related to (1) by Holmes (1977) and Holmes and Marsden (1978). In such cases the dynamical response is much richer. (The first two books under review give accounts of these questions.) One manifestation of this richness is that if the exponents with the largest real parts of a system like (3) consist of a complex conjugate pair and a real one that simultaneously cross the imaginary axis as a parameter λ passes through a critical value, then the corresponding nonlinear system can have chaotic motions.

A canonical example of a Hopf bifurcation is afforded by the first-order system for (x, y) that is converted to

$$(6) \quad \dot{r} = r(\lambda - r), \quad \dot{\theta} = 1$$

by the change of variables $x = r \cos \theta$, $y = r \sin \theta$. The basic ideas of such bifurcations were known to Poincaré and were developed by Andronov and Vitt in the 1930s. In view of this history, E. Hopf (1942) offered his proof of the general theorem with some diffidence. Ruelle and Takens (1971) were apparently the first to use the actual terminology “Hopf bifurcation”. This terminology was employed in the first book completely dedicated to the subject, that of Marsden and McCracken (1976). It is now standard in the West, and is used in the three books under review. Thus, on confronting Hopf bifurcation, the reader who learned ordinary differential equations before the 1970s could at first glance enjoy the same delight as M. Jourdain in discovering that he already knew the subject hiding behind the fancy terminology. A second glance shows that there is more to Hopf bifurcation.

What was Hopf’s specific contribution and how did it influence the modern development of bifurcation theory? Hopf was able to carry out his proof for systems of ordinary differential equations of arbitrary order, when hitherto the analyses were restricted to second-order systems. The basic idea of his proof, that of projecting the solution of the full system of equations onto the span of solutions of a linear second-order system having the characteristic exponents σ with the largest real parts, can be readily adapted to handle nonlinear delay-differential equations and nonlinear parabolic equations. Related bifurcation theories, modelled after Hopf’s, describe the more complicated phenomena, such as bifurcation from periodic to quasiperiodic or almost periodic solutions, that arise when the operator in the linearized equations have more complicated spectral properties. Some of the resulting problems involve small divisors and require very deep analyses.

Many of these effects are manifested by the solutions of the equations describing the flow of a viscous fluid in the annular region between two rotating infinite cylinders. Suppose that the inner cylinder rotates with angular velocity λ and that the outer cylinder is fixed. The equations possess a trivial branch of solutions for all λ , corresponding to the steady *Couette flow* in which each fluid particle has a circular path about the axis. At a critical value of λ , another branch of steady solutions describing Taylor vortices bifurcates from the trivial branch. In this process the column of fluid breaks up into cells in which each fluid particle has a toroidal motion that is a superposition of a circular motion about the axis and a rotation about the circle. At another critical value of λ , the system undergoes a Hopf bifurcation into one in which the cells wobble periodically. (The analysis of this Hopf bifurcation is difficult because the fundamental solution, describing the Taylor vortices, is not known explicitly.) At higher critical values, there are further bifurcations into families of solutions with progressively more complicated behavior until a chaotic state called turbulence is attained. Many rigorous results on these questions have been obtained in the last fifteen years. (For details and references on this

problem consult Rand (1982). For descriptions of related problems, see Joseph (1976.)

The use of the term ‘‘Hopf bifurcation’’ signalizes that bifurcation theory has reached a level of development high enough for the subject to be systematically organized. In such an organization Hopf bifurcation represents the simplest process by which steady solutions bifurcate into unsteady solutions. It is the generic process by which such bifurcations occur for one-parameter problems.

This development is by no means complete, as is evidenced by the problem described by Figure 2. This problem is merely a primitive model for that in which the system of mass points, shafts, and springs is replaced by an elastic column, described by a system of partial differential equations. One expects the usual technical difficulties in raising the dimension of the space of solutions from finite to infinite. But the actual difficulties are more than merely technical: The most accurate nonlinear models for the undamped motion of the column lead to quasilinear hyperbolic systems, which admit shocks. The addition of a very strong dissipative mechanism, which is equivalent to replacing the hyperbolic system by a certain parabolic system of higher order, would eliminate the shocks. But the resulting equations cannot be subsumed under the available versions of the Hopf Bifurcation Theorem for parabolic equation because the operators appearing in these equations do not generate analytic semigroups. These difficulties are magnified when the problem is altered to bring in other kinds of bifurcation. (An inkling of the richness of such open problems can be gleaned from Hermann (1967) and Holmes (1980).)

The three books under review, hereafter denoted respectively by C, HKW, and IJ, are frankly expository treatments of recent results of local bifurcation theory, directed at those interested in applying the theory. As such they attest to the systematization of the subject. From the viewpoint of analysis, the subjects they treat are merely variations on the theme of the contraction mapping principle. But the utilization of this principle often requires great ingenuity and the application of the resulting theory often involves serious computational difficulties (as even a full analysis of (1) would show). Both C and HKW are lecture notes devoted to special but central and important topics in bifurcation theory. IJ is a full-blown text with more ambitious goals.

Center manifolds, the subject of C, are defined for autonomous systems of ordinary differential equations in the following way. Consider the system of the form

$$(7) \quad \dot{x} = Ax + f(x, y), \quad \dot{y} = Bx + g(x, y)$$

where x has values in \mathbf{R}^n , y has values in \mathbf{R}^m , A is a constant matrix all of whose eigenvalues are imaginary, B is a constant matrix all of whose eigenvalues have nonzero real parts, which for simplicity we take to be negative, f and g are twice continuously differentiable, and f, g , and their first derivatives vanish at $(0, 0)$. A basic theorem is that under these conditions there is a twice continuously differentiable function $x \mapsto h(x) \in \mathbf{R}^m$ with h and its first derivatives vanishing at 0 such that $\{(x, y) : y = h(x)\}$ is an invariant manifold for

(7), called a *center manifold*. Moreover, the stability of the solution $(0, 0)$ of (7) is the same as that of the smaller system

$$(8) \quad \dot{x} = Ax + f(x, h(x)).$$

Carr furnishes proofs of these results, but the strength of this book lies in the consistently interesting set of examples with which he exhibits the richness and power of this geometrical version of the implicit function theorem. (Included in these applications are Hopf bifurcation problems and singular perturbation problems.) Five chapters of the book are devoted to ordinary differential equations and one to partial differential equations.

HKW treats the Hopf bifurcation with a view towards applications. It too has a fine collection of examples. It employs center manifolds in its rather lengthy derivation of the theory. Naturally HKW has some overlap with C. The novelty of HKW is its strong orientation toward the numerical implementation of the theory for specific examples. Detailed computer programs, their virtues discussed in the text, are supplied on a microfiche.

IJ has a flavor different from that of the other two books. The authors' primary interest is in the simplest practical methods for studying different kinds of bifurcations. These methods are perturbation methods, which replace the local study of a nonlinear problem by the study of a sequence of nonhomogeneous linear problems. The computational steps involved are greatly simplified by the use of the Fredholm alternative. (The following analogy is apt: In proving the implicit function theorem one can use the contraction mapping principle or, equivalently, show that a certain iteration scheme converges. Once the theorem is proved, however, the local behavior of solutions is most easily found not by implementing the iteration scheme, but by differentiating the original equation as many times as are desired and are permitted, evaluating these derivatives at the base solution, and directly computing the leading terms of the Taylor expansion of the solution. The method of Lyapunov and Schmidt and the center manifold theory are used to reduce a given problem to one to which the contraction mapping principle can be applied.)

IJ treats bifurcation and stability of steady solutions (including one-parameter imperfection theory), Hopf bifurcation, subharmonic bifurcation, and bifurcation into quasiperiodic solutions. There are many examples chosen for their ability to illustrate the theory in a simple way, rather than for their intrinsic interest. Requisite elementary background material from operator theory and from ordinary differential equations is included to make the book accessible to the widest possible audience. But this objective may not be fully attained because of the operator-theoretic character of much of the exposition, a character necessary to exhibit the unity of the authors' approach. Many readers may accordingly prefer to use this book as a reference rather than as a text.

Each of these books is, for the most part, well written. No trouble is caused by an occasional imprecision. (I was irked, however, by the barbarism "parameterize" used in HKW and IJ, by IJ's insistence on classifying periodic

solutions as “equilibrium” solutions, and by IJ’s studied refusal to use “that” as a relative pronoun, a refusal that could make H. W. Fowler undergo the usual rotational instability in his grave.)

Each of these books offers an effective entrée into a lively area of research and a helpful guide for those who wish to apply the theory. Each book would nicely complement the standard texts used in beginning graduate courses in ordinary differential equations.

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STUART S. ANTMAN

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Interpolation of linear operators, by S. G. Kreĭn, Ju. I. Petunin, and E. M. Semenov, Translations of Mathematical Monographs, vol. 54, American Mathematical Society, Providence, R.I., 1982, xii + 375 pp., \$78.00. ISBN 0-6869-1993-9

The pair have reached that fearful chasm,
How tempting to bestride!
For lordly Wharf is there pent in
With rocks on either side.

WILLIAM WORDSWORTH