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THE NEWTON DIAGRAM OF AN ANALYTIC MORPHISM, AND APPLICATIONS TO DIFFERENTIABLE FUNCTIONS

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Consider a system of equations of the form

$$f(x) = A(x) \cdot g(\phi(x)), \tag{1}$$

where $x = (x_1, \ldots, x_m)$, $\phi(x) = (\phi_1(x), \ldots, \phi_n(x))$ is an analytic mapping, and $A(x)$ is a $p \times q$ matrix of analytic functions. Given $f(x) = (f_1(x), \ldots, f_p(x))C^\infty$, we seek C^∞ solutions $g(y) = (g_1(y), \ldots, g_q(y))$. There is a necessary condition on the Taylor series of f at each point. Special cases are classical: when $\phi(x) \equiv x$ we have the division theorem of Malgrange [**7**, Chapter VI], and when $A(x) \equiv I$, the composition problem first studied by Glaeser [**5**].

We solve the problem in the case that $\phi(x)$ and $A(x)$ are algebraic (or Nash), using a Hilbert-Samuel stratification associated to (1). Our methods, however, go far beyond this case. We present algebraic criteria for solving (1), based on a fundamental relationship between two invariants of an analytic morphism and an associated "Newton diagram". Hironaka's simple but powerful formal division algorithm [**3**] is exploited systematically. The only results from "differential analysis" used are Whitney's extension theorem [**7**, Chapter I] and Łojasiewicz's inequality [**7**, Chapter IV].

Let $\underline{k} = \mathbf{R}$ or $\mathbf{C}$. (Some of our assertions hold for other fields.) Let M, N be analytic manifolds (over $\underline{k}$), and $\phi\colon M \to N$ an analytic mapping. Let A be a $p \times q$ matrix of analytic functions on M.

For each $a \in M$, let $\mathcal{O}_a$ (respectively, $\hat{\mathcal{O}}_a$) denote the ring of germs of analytic functions at a (respectively, the completion of $\mathcal{O}_a$ in the Krull topology). Let $\hat{\mathfrak{m}}_a$ be the maximal ideal of $\hat{\mathcal{O}}_a$. In the case $\underline{k} = \mathbf{R}$, let $C^\infty(M)$ denote the algebra of C^∞ functions on M. There is a Taylor series homomorphism $f \mapsto \hat{f}_a$ from $C^\infty(M)^p$ onto $\hat{\mathcal{O}}_a^p$.

The mapping ϕ induces ring homomorphisms $\phi^*\colon C^\infty(N) \to C^\infty(M)$, $\phi_a^*\colon \mathcal{O}_{\phi(a)} \to \mathcal{O}_a$, and $\hat{\phi}_a^*\colon \hat{\mathcal{O}}_{\phi(a)} \to \hat{\mathcal{O}}_a$. Let $\Phi\colon C^\infty(N)^q \to C^\infty(M)^p$ denote the module homomorphism over ϕ^* defined by $\Phi(g) = A \cdot (g \circ \phi)$. Let $\hat{\Phi}_a\colon \hat{\mathcal{O}}_{\phi(a)}^q \to \hat{\mathcal{O}}_a^p$ denote the analogous module homomorphism over $\hat{\phi}_a^*$.

Let $(\Phi C^\infty(N)^q)\hat{}$ denote the $C^\infty(N)$-submodule of $C^\infty(M)^p$ consisting of elements which formally belong to the image $\Phi C^\infty(N)^q$ of Φ; i.e., $(\Phi C^\infty(N)^q)\hat{} = \{f \in C^\infty(M)^p\colon$ for all $b \in \phi(M)$, there exists $G_b \in \hat{\mathcal{O}}_b$ such that $\hat{f}_a = \hat{\Phi}_a(G_b)$ for all $a \in \phi^{-1}(b)\}$. Evidently, $(\Phi C^\infty(N)^q)\hat{}$ is closed in the C^∞ topology.

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THEOREM 1. *Suppose that M, N, ϕ and A are algebraic (or Nash). If ϕ is proper, then* $\Phi C^\infty(N)^q = (\Phi C^\infty(N)^q)\hat{}$.

In the following, we use the same notation for a germ at a point and a representative of the germ in a suitable neighborhood.

Let $s \in \mathbf{N}$. Let $M^s_\phi = \{\mathbf{x} = (x^1, \ldots, x^s) \in M^s \colon \phi(x^1) = \cdots = \phi(x^s)\}$, and let $\phi\colon M^s_\phi \to N$ be the induced morphism.

Invariants of an analytic morphism. Let $\mathbf{a} = (a^1, \ldots, a^s) \in M^s_\phi$. Put $b = \phi(\mathbf{a})$. Let $\mathcal{R}_\mathbf{a}$ denote the submodule $\mathcal{R}_\mathbf{a} = \bigcap_{i=1}^s \operatorname{Ker} \hat{\Phi}_{a^i}$ of $\hat{\mathcal{O}}^q_b$.

By a lemma of Chevalley [4, §II], there is a function $l = l(k, \mathbf{a})$ from $\mathbf{N}$ to itself with the following property: If $G \in \hat{\mathcal{O}}^q_b$ and $\hat{\Phi}_{a^i}(G) \in \hat{\mathfrak{m}}^{l+1}_{a^i} \cdot \hat{\mathcal{O}}^p_{a^i}$, $1 \le i \le s$, then $G \in \mathcal{R}_\mathbf{a} + \hat{\mathfrak{m}}^{k+1}_b \cdot \hat{\mathcal{O}}^q_b$. We say there is a *uniform Chevalley estimate* if we can choose $l = k(k)$ independent of $\mathbf{a}$ (locally in M^s_ϕ).

Let $H_\mathbf{a}(k)$ denote the *Hilbert-Samuel function* of the $\hat{\mathcal{O}}_b$-module $\hat{\mathcal{O}}^q_b/\mathcal{R}_\mathbf{a}$; i.e.,

$$H_\mathbf{a}(k) = \dim_{\underline{k}} \frac{\hat{\mathcal{O}}^q_b}{\mathcal{R}_\mathbf{a} + \hat{\mathfrak{m}}^{k+1}_b \cdot \hat{\mathcal{O}}^q_b}.$$

We say that $H_\mathbf{a}$ is (*analytic*) *Zariski* (*upper-*) *semicontinuous* if, for every irreducible germ of an analytic subset X at a point of M^s_ϕ, there is a germ of a proper analytic subset Y of X, with the following properties:

(i) $H_\mathbf{a}(k)$ is constant in $X - Y$ for all k.

(ii) If $\mathbf{a} \in X - Y$ and $\mathbf{a}' \in Y$, then $H_{\mathbf{a}'}(k) \ge H_\mathbf{a}(k)$ for all k, and $H_{\mathbf{a}'}(k) > H_\mathbf{a}(k)$ for some k.

The Newton diagram. Since all our results are local in N, we assume that N is an open subset of $\underline{k}^n$. Following Hironaka [3], we associate to $\mathcal{R}_\mathbf{a}$ a subset $\mathfrak{N}_\mathbf{a}$ of $\mathbf{N}^n \times \{1, \ldots, q\}$.

If $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$, put $|\beta| = \beta_1 + \cdots + \beta_n$. We order the $(n+2)$-tuples $(\beta_1, \ldots, \beta_n, j, |\beta|)$, where $(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, q\}$, lexicographically from the right. This induces a total ordering of $\mathbf{N}^n \times \{1, \ldots, q\}$.

Let $b = (b_1, \ldots, b_n) \in N$. We identify $\hat{\mathcal{O}}_b$ with the ring of formal power series $\underline{k}[[y - b]] = \underline{k}[[y_1 - b_1, \ldots, y_n - b_n]]$. Let $G \in \hat{\mathcal{O}}^q_b$, $G = (G_1, \ldots, G_q)$. Write $G_j = \sum_{\beta \in \mathbf{N}^n} g_{\beta,j}(y - b)^\beta$, $1 \le j \le q$, where $g_{\beta,j} \in \underline{k}$ and $(y - b)^\beta$ denotes $(y_1 - b_1)^{\beta_1} \cdots (y_n - b_n)^{\beta_n}$. Let $\nu(G)$ denote the smallest (β, j) such that $g_{\beta,j} \ne 0$.

The *Newton diagram* $\mathfrak{N}_\mathbf{a}$ of $\mathcal{R}_\mathbf{a}$ is defined as $\{\nu(G) : G \in \mathcal{R}_\mathbf{a}\}$. Clearly, $\mathfrak{N}_\mathbf{a} + \mathbf{N}^n = \mathfrak{N}_\mathbf{a}$, where addition is defined by $(\beta, j) + \gamma = (\beta + \gamma, j)$, $(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, q\}$, $\gamma \in \mathbf{N}^n$.

REMARKS. (i) There is a smallest finite subset $\mathfrak{V}_\mathbf{a}$ of $\mathfrak{N}_\mathbf{a}$ such that $\mathfrak{N}_\mathbf{a} = \mathfrak{V}_\mathbf{a} + \mathbf{N}^n$.

(ii) $H_\mathbf{a}(k)$ is the number of elements $(\beta, j) \in \mathbf{N}^n \times \{1, \ldots, q\}$ such that $|\beta| \le k$ and $(\beta, j) \notin \mathfrak{N}_\mathbf{a}$.

THEOREM 2. *The following conditions are equivalent:*

(2.1) *There is a uniform Chevalley estimate.*

(2.2) *The Hilbert-Samuel function $H_\mathbf{a}$ is Zariski semicontinuous.*

(2.3) *Each point of* M^s_ϕ *admits a neighborhood* U *and a filtration* $U = X_0 \supset X_1 \supset \cdots \supset X_l = \varnothing$ *by closed analytic subsets of* U, *such that the Newton diagram* $\mathfrak{N}_{\mathbf{a}}$ *is constant on* $X_i - X_{i+1}$ *for each* i.

Suppose $\underline{k} = \mathbf{R}$. If ϕ is proper, then (locally in N) there is a bound s on the number of connected components of a fiber $\phi^{-1}(b)$. With this s, we prove:

THEOREM 3. *Suppose* ϕ *is proper. Then each of the conditions of Theorem 2 implies* $\Phi C^\infty(N)^q = (\Phi C^\infty(N)^q)\hat{}$.

IDEA OF THE PROOF. It follows from Theorem 2 that there is a locally finite partition $\{X_i\}$ of M^s_ϕ such that, for each i:

(i) X_i is a connected smooth semianalytic subset of M^s_ϕ.

(ii) $\overline{X}_i - X_i \subset Y_{i-1}$, where $Y_i = \bigcup_{j\le i} X_j$.

(iii) For all $\mathbf{a} \in X_i - \phi^{-1}(\phi(Y_{i-1}))$, $\mathcal{R}_{\mathbf{a}} = \bigcap_{a\in\phi^{-1}(\phi(\mathbf{a}))} \operatorname{Ker}\hat{\Phi}_a$.

(iv) $\mathfrak{N}_{\mathbf{a}}$ is constant on X_i.

Let $f \in (\Phi C^\infty(N)^q)\hat{}$. By induction on i, we assume f is flat on $\phi^{-1}(\phi(Y_{i-1}))$. Let $\mathbf{a} = (a^1, \ldots, a^s) \in X_i - \phi^{-1}(\phi(Y_{i-1}))$, $b = \phi(\mathbf{a})$. By (iii), (iv) and Hironaka's formal division theorem [**3**], there is a unique $G = G_b \in \hat{\mathcal{O}}_b^q$ such that $\hat{f}_a = \hat{\Phi}_a(G_b)$ for all $a \in \phi^{-1}(b)$, and $g_{\beta,j} = 0$ for all $(\beta, j) \in \mathfrak{N}_{\mathbf{a}}$. Then the G_b are induced by a q-tuple of C^∞ functions which are flat on $\phi(Y_{i-1})$ (cf. [**2**]).

THEOREM 4. *The conditions of Theorem 2 are satisfied in each of the following cases:*

(4.1) M, N, ϕ *and* A *are algebraic.*

(4.2) ϕ *=identity.*

(4.3) $A = I$ *and* ϕ *is* regular; *i.e., for each* $a \in M$, *the Krull dimension of* $\mathcal{O}_{\phi(a)}/\operatorname{Ker}\phi_a^*$ *equals the generic rank of* ϕ *near* a.

(4.4) ϕ *is finite.*

REMARKS. (i) The algebraic hypothesis in (4.1) is essential only to the following point in our proof: If $a \in M$, $b = \phi(a)$, then any $G \in \hat{\mathcal{O}}_b^q$ such that $\hat{\Phi}_a(G) = 0$ can be approximated to any order by an algebraic solution. Writing $y = \phi(x)$ in local coordinates, this amounts to considering the system of equations $A(x)\cdot g(y) = \sum_{i=1}^n h_i(x,y)(y_i - \phi_i(x))$, and finding an algebraic approximation $g(y)$, $h_i(x,y)$ to a given formal solution. Since the equations are linear in the h_i, this special case of "Artin approximation with respect to nested subrings" follows from Artin's theorem [**1**].

(ii) In the classical coherent case (4.2), the uniform Chevalley estimate is equivalent to a "uniform Artin-Rees lemma" (cf. [**9**]), and the conclusion of Theorem 3 is Malgrange's theorem on ideals generated by analytic functions. The uniform Chevalley estimate (2.1) in case (4.3) can be obtained using techniques of [**9**] (as Tougeron showed us); Theorem 3 in this case gives the composition theorem of [**2**]. In case (4.4), condition (2.2) follows from the finite coherence theorem of Grauert and Remmert [**6**], and Theorem 3 recovers a result of Merrien [**8**].

Detailed proofs of our theorems will appear.

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