

ON THE LOCAL LANGLANDS CONJECTURE IN PRIME DIMENSION

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Let F be a local field of residual characteristics p . Then it is a conjecture of Langlands [JL] that there should be a natural bijection between the set of n -dimensional semisimple representations of the absolute Weil-Deligne group of F and the set of irreducible admissible representations of $\mathrm{GL}_n(F)$. Some cases of this conjecture have been established [He, JL, JPS, K, M]. Here we announce further progress toward its verification.

To describe our results, we first note that by work of Bernstein and Zelevinsky [Z], one may restrict one's attention to irreducible representations of the Weil-Deligne group on the one hand and irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ on the other hand. In this context, the conjecture says there should exist a bijection $\sigma \mapsto \pi(\sigma)$ of the set $\mathcal{A}_n^0(F)$ of equivalence classes of continuous, irreducible n -dimensional complex representations of W_F , the absolute Weil group of F , with the set $\mathcal{A}^0(\mathrm{GL}_n(F))$ of equivalence classes of admissible irreducible supercuspidal representations of $\mathrm{GL}_n(F)$. This bijection should satisfy the following conditions:

- (1.01) $\epsilon(\pi(\sigma), \psi) = \epsilon(\sigma, \psi)$ (see [D, GJ] for definitions),
- (1.02) $\pi(\sigma) \otimes \chi \circ \det = \pi(\sigma \otimes \chi)$ for all quasi-characters χ of F^\times ,
- (1.03) $\omega_{\pi(\sigma)} = \det \sigma$, where $\omega_{\pi(\sigma)}$ is the central character of $\pi(\sigma)$.

We note that if $n = 1$, the existence of such a bijection is a restatement of the fundamental theorem of local class field theory [S]; thus when $n \geq 2$, the conjecture under consideration may be thought of as a nonabelian analogue of that theorem.

When $n \geq 2$ the construction of $\pi(\sigma)$ breaks naturally into two steps.

I. *Construction of $\pi(\sigma)$ when σ is induced from a representation of smaller dimension.* This construction is provided when $n = 2$ by decomposing the Weil representation of $\mathrm{SL}_2(F)$ (see [JL]). When $n = 3$, it is obtained by global methods [JPS]. When $p \nmid n$ then all n -dimensional irreducible representations σ of W_F are monomial, and one may use a representation which induces σ to construct a supercuspidal representation $\pi'(\sigma)$ of $\mathrm{GL}_n(F)$. This was first done by Howe [Ho], who conjectured that $\pi'(\sigma)$ satisfied (1.01)–(1.03). Recently, Moy [M] showed that a representation $\pi(\sigma)$ satisfying (1.01)–(1.03) may be obtained by a slight modification of Howe's construction and thus verified the Langlands conjecture in case $p \nmid n$ (one needs, however, that $\mathrm{char} F = 0$ in order that the map $\sigma \mapsto \pi(\sigma)$ be bijective).

When $p \mid n$, however, the above approach appears to fail.

Received by the editors June 27, 1983.

1980 *Mathematics Subject Classification.* Primary 12B25, 20G05; Secondary 22E50, 12B15.

¹Supported in part by NSF Grant #MP575 07481.

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0273-0979/83 \$1.00 + \$.25 per page

II. *Extension of the map $\sigma \rightarrow \pi(\sigma)$ to primitive representations σ .* Such an extension was obtained in case $n = p = 2$ by Kutzko in [K] and in case $n = p = 3$ by Henniart in [He], thus verifying the Langlands conjecture in these cases.

In general, one has reason to expect that step I above may be approached using global methods and that, in particular, such methods will lead to the construction of $\pi(\sigma)$ when σ is induced from a one-dimensional representation on a normal subgroup of W_F . We here announce that, given such a global construction for monomial representations σ of dimension $n = p$, we are able to construct $\pi(\sigma)$ for an arbitrary p -dimensional representation σ . To be precise, we must make the following

ASSUMPTION 1.1. Let K be a local field of residual characteristic p and let E/K be a cyclic extension of degree p . Let θ be a quasi-character of E^\times which is not fixed by the galois group, $\Gamma_{E/K}$, of E/K and let $\sigma = \text{Ind}_{W_E \uparrow W_K} \theta$. Then there is an irreducible supercuspidal representation $\pi(\sigma)$ of $\text{GL}_p(K)$ which satisfies (1.01)–(1.03). Furthermore, the map $\sigma \rightarrow \pi(\sigma)$ is injective, and if L is a field over which K is galois, then the map $\sigma \rightarrow \pi(\sigma)$ commutes with $\Gamma_{K/L}$.

We then prove

THEOREM 1.2. *Suppose Assumption 1.1 holds for all extensions K/F . Then, given any irreducible p -dimensional representation σ of W_F , there exists an irreducible supercuspidal representation $\pi(\sigma)$ of $\text{GL}_p(F)$ satisfying (1.01) – (1.03).*

We proceed as follows (proofs will be provided elsewhere).

DEFINITION 1.3. Call a galois extension K/F a *good extension* of F , either if $\Gamma_{K/F}$ is cyclic and either $[K : F] \mid p - 1$ or $[K : F] \mid p + 1$, or if $\Gamma_{K/F}$ is a generalized quaternion group and $[K : F] \mid 2(p + 1)$.

PROPOSITION 1.4. *Let K/F be a good extension. Then there is a map $\text{lift}_{K/F}$ which maps irreducible supercuspidal representations of $\text{GL}_p(F)$ to irreducible supercuspidal representations of $\text{GL}_p(K)$ and has the following properties:*

(1.4.1) *If π is an irreducible supercuspidal representation of F , then $\text{lift}_{K/F} \pi$ is fixed under $\Gamma_{K/F}$;*

(1.4.2) $\epsilon(\text{lift}_{K/F} \pi, \psi \circ \text{tr}_{K/F}) = [\epsilon(\pi, \psi)]^{[K:F]} \lambda_{K/F}^{-p} \cdot \delta(\pi)$, *where $\lambda_{K/F}$ is the Langlands constant associated to K/F and $\delta(\pi) = \pm 1$;*

(1.4.3) $\text{lift}_{K/F}(\pi \otimes \chi) = \text{lift}_{K/F}(\pi) \otimes \chi \circ N_{K/F}$ *for any quasi-character χ of F^\times ;*

(1.4.4) *Let π_K be an irreducible supercuspidal representatin of K which is fixed under $\Gamma_{K/F}$ and let $\omega(\pi_K)$ be its central character. Then if ω is a quasi-character of F for which $\omega_K = \omega \circ N_{K/F}$, there is exactly one irreducible supercuspidal representation π of $\text{GL}_p(F)$ for which $\omega(\pi) = \omega$ and $\text{lift}_{K/F} \pi = \pi_K$.*

PROPOSITION 1.5. *Let σ be an irreducible p -dimensional representation of Γ_F . Then there is a good extension K/F , a cyclic extension E/F and a quasi-character θ of E^\times such that $\sigma|_{W_K} = \text{Ind}_{W_E \uparrow W_K} \theta$. Furthermore,*

$$\epsilon(\sigma|_{W_K}, \psi_{K/F}) = [\epsilon(\sigma, \psi)]^{[K:F]} \lambda_{K/F}^{-p} \delta(\sigma), \quad \text{where } \delta(\sigma) = \pm 1.$$

PROPOSITION 1.6. *With notation as above, and assuming 1.1, define the representation $\pi(\sigma)$ of $\mathrm{GL}_p(F)$ by the conditions $\mathrm{lift}_{K/F}\pi(\sigma) = \pi(\sigma|_{W_K})$, $\omega(\pi(\sigma)) = \det \sigma$. Then $\epsilon(\pi(\sigma), \psi) = \xi \epsilon(\sigma, \psi)$, where ξ is a p th-power root of unity.*

PROPOSITION 1.7. *$\pi(\sigma)$ satisfies conditions (1.01) – (1.03). If the map $\sigma \mapsto \pi(\sigma)$ given by Assumption 1.1 has the additional property that it commutes with $\mathrm{lift}_{K/F}$, then the map $\sigma \mapsto \pi(\sigma)$ constructed above is unique and injective. If all irreducible supercuspidal representations of $\mathrm{GL}_p(F)$ may be constructed by induction from open compact-modulo center subgroups (see [C]), then the map $\sigma \mapsto \pi(\sigma)$ is a bijection of the set of equivalence classes of irreducible p -dimensional representations of $W(F)$ with the set of equivalence classes of irreducible supercuspidal representations of $\mathrm{GL}_p(F)$ (see also [Ko]).*

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