

INVARIANTS OF FORMAL GROUP LAW ACTIONS¹

BY ROBERT M. FOSSUM

0. Introduction. In this note, k denotes a field of characteristic $p > 0$, and the letters T , X and Y are formal indeterminants. Let $F: k[[T]] \rightarrow k[[X, Y]]$ be a (fixed) one-dimensional formal group law [Dieudonné, Hazewinkel, Lazard, Lubin] of height $h \geq 0$. Let V denote a $k[[T]]$ module of finite length. Suppose $\text{Ann}(V) = (T^n)$. Let $q = p^e$ denote the least power of p such that $n \leq q$. It follows that the symmetric powers $S_r(V)$ over k become $k[[T]]$ -modules, annihilated by T^q , through the formal group law, viz: If $F(T) = X + Y + \sum_{i,j \geq 1} C_{ij} X^i Y^j$ and f is in $S_t(V)$ and g is in $S_s(V)$, then

$$T(fg) = fTg + (Tf)g + \sum C_{ij}(T^i f)(T^j g)$$

in $S_{t+s}(V)$.

Denote by $S.(V)$ the symmetric algebra on V ; so

$$S.(V) := \bigoplus_{r > 0} S_r(V).$$

Then $S.(V)$ is a $k[[T]]$ -module annihilated by T^q . The main purpose of this note is to announce and outline a proof of the theorem below. Several consequences and examples are included.

THEOREM. *Let $S.(V)^F := \{f \in S.(V) : Tf = 0\}$. The set $S.(V)^F$ is a normal noetherian subring of $S.(V)$ of the same Krull dimension. Furthermore, $S.(V)^F$ is factorial.*

1. An outline of the proof. To prove the Theorem one can consider two cases: $\text{ht } F = h = 1$ and $\text{ht } F = h \neq 1$. In case $\text{ht } F = 1$, the action on $S.(V)$ is equivalent to an action of the cyclic group $\mathbf{Z}/q\mathbf{Z}$ on $S.(V)$. This case is considered, in full generality, in [Fossium, Griffith] and [Almkvist, Fossium].

So consider the case $\text{ht } F \neq 1$. It can be shown that there is a fixed power s of p , depending only on $\text{ht } F$, such that $S.(V)^s \subset S.(V)^F$. Then one can extend the action of T to the field of fractions L of $S.(V)$ via

$$T(f/g) = T(fg^{s-1})/g^s.$$

Then one concludes that L^F is a field and

$$S.(V)^F = L^F \cap S.(V),$$

which shows that $S.(V)^F$ is a Krull domain and $S.(V)^F \supset k[S.(V)^s]$, which shows that $S.(V)^F$ is noetherian and $S.(V)$ is integral over $S.(V)^F$. Hence,

Received by the editors May 2, 1983.

1980 *Mathematics Subject Classification.* Primary 14L05, 14L30, 13H10, 13F15.

¹This research has been supported by the National Science Foundation.

© 1983 American Mathematical Society
 0273-0979/83 \$1.00 + \$.25 per page

the extension $S.(V)^F \rightarrow S.(V)$ is (PDE) (cf. [Fossum]). It remains to prove that $S.(V)^F$ is factorial. As $S.(V)$ and $S.(V)^F$ are graded, it is sufficient to consider homogeneous ideals and to show that each homogeneous, prime, divisorial ideal \mathfrak{p} in $S.(V)^F$ is principal. One could accomplish this by using results in [Waterhouse], but there is also a straightforward calculation.

Suppose \mathfrak{p} is a homogeneous prime divisorial ideal in $S.(V)^F$. Let $(\mathfrak{p}S.(V))^{**}$ denote the divisorial ideal it generates in $S.(V)$. This ideal is principal, generated by a homogeneous element f in $S.(V)$. It follows that

$$f^{-1}S.(V) = \{x \in L : xp \subset S.(V)\}.$$

Since $f^{-1}\mathfrak{p} \subset S.(V)$ and $T(\mathfrak{p}) = 0$, it follows easily that $T(f^{-1})\mathfrak{p} \subset S.(V)$. Hence, there is an element $a \in S.(V)$ such that

$$T(f^{-1}) = f^{-1}a, \text{ or } T(f^{s-1}) = f^{s-1}a.$$

Since degree $T(f^{s-1}) = \text{degree } f^{s-1}$, it follows that degree $a = 0$, or that $a \in k$. Hence $T(a) = 0$. Then

$$T^r(f^{s-1}) = f^{s-1}a^r.$$

Since $T^q = 0$, one obtains that $a^q = 0$ or $a = 0$. Hence, $T(f^{s-1}) = 0$. Then

$$0 = T(f^s) = fT(f^{s-1}) + T(f)f^{s-1} = T(f)f^{s-1}.$$

Hence, $T(f) = 0$, so $f \in S.(V)^F$. From this one concludes that $\mathfrak{p} = fS.(V)^F$. This concludes the (outline) of the proof. Details that are omitted will appear elsewhere.

2. EXAMPLES. For each $n \in \mathbb{N}$, set $V_n := k[[T]]/(T^n)$. The Hilbert series of $S.(V_n)^F$ is, by definition, the power series

$$\sum_{r=0}^{\infty} (\text{rk}_k S_r(V_n)^F) t^r \in \mathbb{Z}[[t]].$$

Denote this series by $H_t(V_n, F)$. (The dependence on p is implicit.) In case $F(T) = X + Y + XY$ (and so $\text{ht } F = 1$) these Hilbert series have been studied extensively in [Almkvist, Fossum]. In particular, for $n = q = p^e$,

$$H_t(V_{p^e}, F) = p^{-e} \left\{ (1-t)^{-p^e} + \sum_{j=1}^e (p^j - p^{j-1})(1-t^{p^j})^{-p^{e-j}} \right\}.$$

Some results are now available for the additive formal group law $F(T) = X + Y$ (so $\text{ht } F = 0$). In this case, for $q = p^e$,

$$\begin{aligned} H_t(V_q, X + Y) &= q^{-1} \{ (1-t)^{-q} + (q-1)(1-t^p)^{-q/p} \\ &= (1-t^p)^{-q/p} + q^{-1} \{ (1-t)^{-q} - (1-t^p)^{-q/p} \}. \end{aligned}$$

This can be used to show that

$$H_t(V_{q-1}, X + Y) = (1-t^p)^{-q/p} + (1-t)q^{-1} \{ (1-t)^{-q} - (1-t^p)^{-q/p} \}.$$

This rational function of t is not unimodal for most q , and hence $S.(V_{q-1})^{(X+Y)}$ is not Gorenstein for these q . Thus $S.(V_{q-1})^{(X+Y)}$ is not Cohen-Macaulay. Evidence suggests that $S.(V_n)^F$ is not Cohen-Macaulay for $n \geq 4$, except in

case $p = 2$ and $n = 4$ (cf. [Stanley]). The case $p = 2$ has been studied by [Bertin]. The Hilbert series

$$H_t(V_4, X + Y + XY) = \frac{1 - t + t^2 + t^3}{(1-t)^2(1-t^2)^2(1+t^2)},$$

which shows that $S.(V_4)^{X+Y+XY}$ is not Cohen-Macaulay, since it is factorial and not Gorenstein.

The Hilbert series, $p = 2$, for V_4 is

$$H_t(V_4, X + Y) = \frac{(1+t^3)}{(1-t)(1-t^2)^3}.$$

A calculation shows, for $S.(V_4) = k[X_3, X_2, X_1, X_0]$ with $TX_3 = X_2$, $TX_2 = X_1$, $TX_1 = X_0$, $TX_0 = 0$, and $T(fg) = f(Tg) + (Tf)g$, that

$$S.(V_4)^{X+Y} = k[X_0, X_1^2, X_2^2, X_3^2, X_3X_0^2 + X_2X_1X_0 + X_1^3],$$

which is a complete intersection.

Further results will appear in more detail.

REFERENCES

- [G. Almkvist and R. Fossum] *Decomposition of exterior and symmetric powers of indecomposable $\mathbb{Z}/p\mathbb{Z}$ -modules in characteristic p and relations to invariants*, Sémin. d'Algèbre Paul Dubreil, M.-P. Malliavin, Lecture Notes in Math., No. 641, Springer-Verlag, Berlin and New York, 1978, pp. 1–111.
- [M.-J. Bertin] *Anneaux d'invariants d'anneaux de polynômes, en caractéristique p* , C. R. Acad. Sci. Paris **264** (1967), 653–656.
- [Jean Dieudonné] *Introduction to the theory of formal groups*, Dekker, New York, 1973.
- [Robert Fossum] *The divisor class group of a Krull domain*, Ergeb. Math. Grenzgeb., Bd. 74, Springer-Verlag, Berlin and New York, 1973.
- [R. Fossum and P. A. Griffith] *Complete local factorial rings which are not Cohen-Macaulay in characteristic p* , Ann. Sci. École Norm. Sup. (4) **8** (1975), 189–200.
- [Michael Hasewinkel] *Formal groups and applications*, Academic Press, New York, 1978.
- [Jonathan Lubin] *Introduction à la théorie des groupes formels*, Secrétariat Math. École Norm. Sup. Paris, 1970.
- [Michel Lasard] *Commutative formal groups*, Lecture Notes in Math., No. 443, Springer-Verlag, Berlin and New York, 1975.
- [Richard Stanley] *Hilbert functions of graded algebras*, Adv. Math. **28** (1978), 57–83.
- [William Waterhouse] *Divisor classes in pseudo-Galois extensions*, Pacific J. Math. **36** (1971), 541–548.

INSTITUT FOR ALGEBRAIC MEDITATION AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801

