BOOK REVIEWS

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The core model, by A. J. Dodd, London Mathematical Society Lecture Notes Series No. 61, Cambridge Univ. Press, New York, New York, 1982, xxxviii + 229 pp., $24.95. ISBN 0-5212-8530-5

Measurable cardinals have been around for a long time. Stanislaw Ulam [**U**] invented them in his salad days in Lwów, and they would emerge from time to time under various combinatorial guises. But they seemed at best a curiosity in a sideshow to the creative achievements of Kurt Gödel in mathematical logic. The driving life force in set theory at the time was the Continuum Problem, and Gödel [**Gö**] in 1938 established the consistency of Cantor's Continuum Hypothesis and also the Axiom of Choice by constructing the universe L of constructible sets. The reasons are not altogether clear for the prolonged lull that ensued, but at least during this period, the structural approach to *restrictions* of models of set theory initiated by Gödel's beautiful construction was systematized in the study of inner models (Sheperdson) and relative constructibility (Lévy). Then in 1963, the analyst Paul Cohen [**C**] established the independence of the Continuum Hypothesis and the Axiom of Choice. In his Forcing Method, the brilliant carpetbagger happened upon a remarkably fecund method for producing *extensions* of models of set theory. There was no lull this time, as fine mathematicians like Robert Solovay quickly perceived the possibilities abounding, and within a few years the general theory of Forcing was codified, and a cornucopia of relative consistency results were being fashioned.

This mainstream of new vitality in set theory was fed by another development, which is traced in more detail to establish the context for the book under review. In 1960, Dana Scott [**Sc**] proved that if there is a measurable cardinal, then the universe V of all sets is strictly larger than Gödel's constructible universe L, i.e. $V \neq L$. The usefulness of the ultraproduct construction in model theory was just beginning to be understood at the time when Scott struck on the idea of taking an ultrapower of the entire universe V. Not only did this simple but penetrating result firmly establish that new axioms can decide outstanding questions about the universe, but it quickly led to the intrinsic characterization of measurable cardinals as critical points (least ordinal moved) of elementary embeddings $j\colon V \to M$ of the universe V into some inner model M. If the work of Gödel and others had established a sacred tradition in logical syntax and formal structure, then Scott's result rescued

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measurable cardinals from the profane, for over the algebraic considerations of the complementary techniques of forcing and inner models was superposed the categorical imperative of embeddings.

Striking results exploiting model-theoretic techniques quickly followed. Frederick Rowbottom **[R]** established that a simple combinatorial property of measurable cardinals already implies that there are only countably many reals in L. By 1965, Jack Silver **[Si2]** (see also Solovay **[So]**) distilled the essence of this transcendence over L by isolating a set of integers called $0^{\#}$. $0^{\#}$ codes all the sentences true of L, but much more than that, it is a veritable blueprint with complete genetic information for the uniform generation of L. If Gödel's breakthrough in the formulation of L over type theory was to generate over the class of ordinals as impredicatively given, the presence of $0^{\#}$ amounts to a trivial crystalization of this process. The story of $0^{\#}$ is a singular example of the interplay of syntax and structure in the isolation of a new axiomatic principle.

Another outgrowth of Scott's result was the investigation of iterated ultrapowers. Haim Gaifman **[Ga]** initiated this study and provided a general theory of "self-extension" operators, but in the context of measurability it was Kenneth Kunen **[K]** who, by 1968, had refined Gaifman's technique to provide an elegant and powerful method for reproving Silver's results as well as deriving structure theorems about inner models of measurability. One of his insights was that ultrapowers can be taken even when the corresponding ultrafilter is not in the model, and this, for example, led to the intrinsic characterization of $0^{\#}$: it exists just in case there is an elementary embedding $j\colon L \to L$. Turning to inner models of measurability, if U is an ultrafilter attesting to the measurability of a cardinal κ, $L[U]$ is the smallest inner model in which κ remains measurable. Silver **[Si1]** had established that $L[U]$ has coarse structural properties like L which made it worth investigating further, and Kunen got to the heart of the matter by demonstrating that the models $L[U]$ depend only on κ (not U) and that all such models are iterated ultrapowers one to the next. Thus, iterated ultrapowers became an intrinsic feature of measurability.

Around this time, there was growing interest in the so-called Singular Cardinals Problem: for singular cardinals κ, what constraints are imposed on the size of 2^{κ} by the sizes of 2^{α} for $\alpha < \kappa$? This seemed to be just a difficult Forcing problem until Silver **[Si3]**, in the summer of 1974, demonstrated that there were definite constraints when κ had uncountable cofinality. Silver's argument was simple, as Scott's was, but it revised the collective intuition of set theorists and spurred new activity. This culminated in Ronald Jensen's Covering Theorem **[DeJ]**, easily the most significant result of the 1970s in set theory: $0^{\#}$ does not exist only when V has the *covering property* with respect to L, i.e., whenever X is an uncountable set of ordinals, there is a $Y \in L$ such that $X \subseteq Y$ and $|X| = |Y|$. This is a deep structural statement about the proximity of V to L, and has as a simple consequence that in the absence of $0^{\#}$, for any singular κ, 2^{κ} must always be the least possible cardinal allowed by classical cardinal arithmetic. That such an ostensibly minimal hypothesis as the violation of the covering property could already lead to $0^{\#}$ was quite revelatory.

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It was against this potent backdrop that the Core Model K was conceived. The speculation that several of the beautiful results about measurable cardinals arose out of proposals, some modest others not, for demonstrating the outright inconsistency of measurable cardinals is particularly pertinent for the Core Model. Measurable cardinals have proved very resilient, and even though they may yet be pursued by a dogged few, there is a general concensus that they do not lead to inconsistency. Whatever the initial incentives, the formulation of the Core Model is a considerable achievement which synthesizes much that came before. It is a definable inner model in which there is no measurable cardinal, yet is rigid with respect to the process of adjoining local endomorphic embeddings (or rather, their characterizing coding sets, the "sharps"). It allows an extension of the Covering Theorem to the consistency strength of measurability, and it provides a fully adequate structural understanding of the gap between $0^{\#}$ and measurability. The Core Model is the result of a close collaboration between Anthony Dodd and Ronald Jensen. Beyond their joint articles [**DoJ1, DoJ2, DoJ3**], the book under review is the full exposition on the subject by Dodd. It is a formidable work, well-organized and self-contained, with the intricate details painstakingly written out.

The extended introduction provides a panoramic overview which manages to touch upon many of the delicate but critical points. It is very well written and well worth reading. The reader is gently eased into the text and is handed a helpful guide for the rigors ahead.

Part I deals with fine structure. Fine structure theory, the stuff of projecta, the remarkable Σ_n uniformization theorems and master codes, is the invention of Jensen [**J**] in his deep analysis of L. The author develops a generalized version for constructibility relative to a predicate, introducing the concepts of *acceptability* (a closure condition prefigured in Silver's proof [**Si1**] of the GCH in $L[U]$) and *soundness* (that the Skolem hull of the projectum plus the coding parameter is the full structure) necessary to effect a fine structure analysis. This part is a veritable paean to formalism as well as a testament to the author's tenacity. In strict adherence to the principle of parsimony, a minimal base set theory is adopted to carry out a completely syntactic study, and further axioms are adjoined only when needed. In doing this, the author anticipates a criticism which should nonetheless be made: It does not seem necessary to pursue such an abstract course when pragmatic assumptions, such as the well-foundedness of structures, would suffice for the text and probably all future work. There may be no substitute for honest toil, but there is no need for gratuitous suffering. One gets a queasy feeling when so many details are presented, and some are seemingly left out, as if something may be rotten at the core. (But perhaps I am not the one to judge, since I get a similar feeling whenever I ponder the details of Gödel's Second Theorem!)

Part II deals with iterated ultrapowers. The elegant theory of Kunen [**K**], especially the indiscernibility of critical points and the iterability criteria, is carefully tailored to a restricted context. Here are presented the basic modules called *iterable premice*, local structures J_α^U which satisfy "U is a normal ultrafilter over κ" for some $\kappa < \alpha$ and yield well-founded iterated ultrapowers.

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Part III synthesizes the two previous parts, developing the remarkable connection between fine structure and iterated ultrapowers first enunciated in some unpublished work of Solovay: Suppose that $N = J_\alpha^U$ is an iterable premouse with measurable κ. Let ρ_N^n be the Σ_n-projectum of N, and X the Skolem hull of ρ_N^1 together with a parameter defining some offending set $z \subseteq \rho_N^1$ which is Σ_1 over N but not a member of N. If π: $X \to M$ is the Mostowski collapse of X, then M is an iterable premouse, and if $\pi(\kappa) < \kappa$, we can take its κth iterated ultrapower M_κ. M_κ turns out to be J_β^U for some β. However, if $\rho_N^1 \leqslant \kappa$, then $\alpha = \beta$, roughly since that set z is below critical points and, hence, preserved, yet is not supposed to be a member of N. Thus, when $\rho_N^1 \leqslant \kappa$, N is an iterated ultrapower of M, and M is called the *core* of N. This is perhaps the ultimate manifestation of the close interaction between definability and iterated ultrapowers, first educed by Kunen in his work on $L[U]$.

With fine structure we can generalize this to Σ_n for $n > 1$. Notice first of all that if $\rho_N^n > \kappa$ for every n, then every N-definable subset of κ is in N. To add such N indiscriminately to the Core Model amounts to introducing measurable cardinals, so it is avoided. So, assume there is some crossing of κ, $\rho_N^{n+1} \leqslant \kappa < \rho_N^n$. We can consider $J_{\rho_N^n}^A$, where A is a Σ_n-master code for N, find its core, and extend embeddings back up to the level of N. This whole analysis involves considerable complications rigorously carried out by the author in the text. When iterations of these extended embeddings at the level of N are all well-founded, N is called a *mouse*. Notice that the very formulation of mice essentially involves fine structure. Part III concludes with an extended study of mice and points out that $0^\#$ exists *iff* there is an iterable premouse *iff* there is a mouse N with $\rho_N^1 = \kappa$ (where κ is the measurable cardinal of N).

In Part IV we are at last treated to our first glimpse of the Core Model K. The toils of the ascent have left the formulation very simple: K is the union of L together with all mice. The author verifies that K models ZFC + GCH, and notes its dependence on the assumptions made in the universe: if $V = L$, then $K = L$; if $0^\#$ exists but there is no j: $L[0^\#] \to L[0^\#]$, then $K = L[0^\#]$, etc.; at the other end, if there is an inner model with a measurable cardinal, then K is the intersection of all its iterated ultrapowers. Nonetheless, the natural ramifications of K look rather murky when compared to the L_α-ramification of L, so the author sharpens the focus with his "sharplike" mice, and the demonstration that if there is a nontrivial j: $K \to K$, then there is an inner model with a measurable cardinal. Part IV is brought to an end with the pleasing results about K being the natural inner model for Ramsey cardinals and, in general, the Erdös cardinals.

The primary motivation for the formulation of the Core Model was to extend Jensen's Covering Theorem. Part V is devoted to generalizations of Covering of which the most accessible is: If there is no inner model with a measurable cardinal, then V has the covering property with respect to K, i.e. whenever X is an uncountable set of ordinals, there is a $Y \in K$ such that $X \subseteq Y$ and $|X| = |Y|$. A necessarily more complicated version is also derived for $L[U]$. After the relative calm of the previous part, the book amplifies to its highest level of structural complexity, as the elements of Jensen's fine structure

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proof for L are again trotted out. In a substantially more Byzantine guise, we behold the essential idea of the upward extension of embeddings, as well as the "vicious" sequences. Probably, the use of these sequences will in the long run be considered a red herring, given Silver's simpler proof of Covering, which needs no fine structure and uses a more elegant device to take care of purported ill-foundedness (see Holzman **[H]**). The author is well aware of the Silver proof, but persists with the exposition of the original road to discovery, averring that the fine structure of K is of intrinsic interest and opining that the techniques involved might be crucial in future work.

This brings to mind the whole question of the role of fine structure. The author addresses himself to this issue at some length in the introduction and elsewhere. Certainly, K can be fully developed without fine structure (defined as the union of L together with all iterable premice N with measurable κ such that $\rho_N^1 \leqslant \kappa$), and arguably, much of the fine structure results deal with its own paraphernalia. All the major results like Covering can be established with not much more than Condensation and Σ_n-substructures, and proofs employing only such elementary techniques are much more accessible to a wide audience. But undeniably, this fine research was first carried out with fine structure, and there is a long series of new principles and results first formulated through the fine structural analysis of constructibility. Perhaps we can best say that the road to discovery is inseparable from the discoverers, that the insights and directions of research of Jensen and his collaborators have been just as crucial as the formalism of fine structure which came most naturally to them. The author clearly feels that the information provided by fine structure is very important, and the book is just as much an exposition of its techniques as the results it establishes.

The concluding Part VI is open-ended and speculative, sketching possible generalizations of K. In 1973, Mitchell **[M1]** had investigated strong forms of measurability, now called measurable cardinals of high Mitchell order, providing internal definitions in terms of *coherent sequences* $\vec{U}$ of ultrafilters and establishing that the corresponding inner models $L[\vec{U}]$ share many properties with inner models of just measurability. After the appearance of K, Mitchell **[M2]** quickly developed corresponding Core Models $K[\vec{U}]$ for coherent sequences $\vec{U}$ and even established some weak versions of Covering. The efficacy of Mitchell's methods is evidenced by his very recent clarification of the Singular Cardinals Problem: he has relative consistency, almost equiconsistency, results concerning various forms of singular strong limit cardinals κ such that $2^\kappa > \kappa^+$, all of which hover around the existence of measurable cardinals κ of Mitchell order κ, κ^+, and κ^{++}. This considerably resolves the author's speculations concerning Covering and powers of singular cardinals, but somewhat to his vindication, Mitchell too first used fine structure and has not been able to eliminate it altogether.

This impressive work of Mitchell suggests that the limits of Core Model technology may have been reached. Much less is known about possible inner models for large cardinals. Mitchell **[M3]** developed his *hypermeasurable cardinals* for this ascent, and the author presents an equivalent treatment in terms of

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his *extenders* and *strong cardinals*. Strong cardinals are a weak version of the well-known supercompact cardinals, and the author even speculates on *superstrong cardinals*, which are the analogously weak versions of the huge cardinals. Much of the overlay of complexity throughout the book is intended to set the stage for the fine structure theory of extenders. Sometimes this complexity is apparently only notational, as in the case of the author's reformulation of Solovay's already clear method for extracting Σ_n-indiscernibles from Σ_1-indiscernibles (pp. 56–58). In fact, remarks peppered throughout the book generate a definite anticipatory air, as we become aware of the author's predilection for presenting general frameworks whenever possible. So, it is disappointing that we are not shown the details of how these best-laid plans are brought to bear on the development of the fine structure of extenders. But, this is presumably the subject of a sequel.

We have come a long way from those halcyon days in the coffee houses of Lwów. *The Core Model* will not be the ultimate exposition on its results, but everything is there, suffused with the fine structure intuition of one of its primary developers. The canon has now been firmly established for future revisionism, and it is up to detractors of formalism and fine structure to provide the counterpoint.

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Topics in iteration theory, by György Targonski, Studia Mathematica, Skript 6, Vandenhoeck & Ruprecht, Gottingen and Zurich, 1981, 292 pp., (kart. DM 45, --), ISBN 3-5244-0126-9

> Falstaff, to Prince Hal: "Oh, thou hast damnable iteration, and art indeed able to corrupt a saint."
>
> Henry IV, Part 1: Act 1, Scene 2

As the quotation shows, iteration, in the general sense of repetition of an act, has been around a long time. Even as a mathematical discipline devoted to the study of the repeated composition of functions with themselves, iteration theory is rather old: it can be said to have begun with the activities of the Cambridge Analytical Society (Babbage, Herschel, Peacock) in the 1810s, and more particularly with the publication by Charles Babbage of his two-part *Essay towards the calculus of functions* in the Philosophical Transactions of the Royal Society in 1815 and 1816.

In that essay, as elsewhere, Babbage wrote ψ^n for the nth iterate of the function ψ, and posed the problem "Required the solution of

$$\psi^n x = x \cdots \text{''}. \tag{1}$$

He observed that if ψ_1 is a solution of (1) and ψ_2 is defined by

$$\psi_2 = f^{-1} \circ \psi_1 \circ f, \tag{2}$$

where f is any invertible function whose range includes the domain and range of ψ_1, then ψ_2 is also a solution of (1). Thus Babbage introduced the equivalence relation of *conjugacy of functions*. Conjugacy is a fundamental notion in iteration theory, for it is clear from (2) that all information about the iterative behavior of a function can be obtained from the corresponding behavior of any conjugate function. For example, for $0 < \lambda \leqslant 2$, let g_λ be the function defined on [0, 1] by

$$g_\lambda(x) = 2\lambda x(1-x).$$

Then g_λ is conjugate to the function h_λ defined on $[-\lambda, \lambda]$ by

$$h_\lambda(x) = x^2 - \lambda(\lambda - 1)$$

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