[Si3] ______, *On the singular cardinals problem*, Proc. Internat. Congress. Math., Vancouver, 1974, pp. 265–268.

[So] Robert Solovay, *A nonconstructible Δ^1_3 set of integers*, Trans. Amer. Math. Soc. **127** (1967), 50–75.

[U] Stanislaw Ulam, *Zur masstheorie in der algemein Mengenlehre*, Fund. Math. **16** (1930), 140–150.

AKIHIRO KANAMORI

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 3, November 1983

0273-0979/83 $1.00 + $.25 per page

Topics in iteration theory, by György Targonski, Studia Mathematica, Skript 6, Vandenhoeck & Ruprecht, Gottingen and Zurich, 1981, 292 pp., (kart. DM 45, --), ISBN 3-5244-0126-9

> Falstaff, to Prince Hal: "Oh, thou hast damnable iteration, and art indeed able to corrupt a saint."
>
> Henry IV, Part 1: Act 1, Scene 2

As the quotation shows, iteration, in the general sense of repetition of an act, has been around a long time. Even as a mathematical discipline devoted to the study of the repeated composition of functions with themselves, iteration theory is rather old: it can be said to have begun with the activities of the Cambridge Analytical Society (Babbage, Herschel, Peacock) in the 1810s, and more particularly with the publication by Charles Babbage of his two-part *Essay towards the calculus of functions* in the Philosophical Transactions of the Royal Society in 1815 and 1816.

In that essay, as elsewhere, Babbage wrote ψ^n for the nth iterate of the function ψ, and posed the problem "Required the solution of

$$\psi^n x = x \cdots \text{''}. \tag{1}$$

He observed that if ψ_1 is a solution of (1) and ψ_2 is defined by

$$\psi_2 = f^{-1} \circ \psi_1 \circ f, \tag{2}$$

where f is any invertible function whose range includes the domain and range of ψ_1, then ψ_2 is also a solution of (1). Thus Babbage introduced the equivalence relation of *conjugacy of functions*. Conjugacy is a fundamental notion in iteration theory, for it is clear from (2) that all information about the iterative behavior of a function can be obtained from the corresponding behavior of any conjugate function. For example, for $0 < \lambda \leq 2$, let g_λ be the function defined on [0, 1] by

$$g_\lambda(x) = 2\lambda x(1 - x).$$

Then g_λ is conjugate to the function h_λ defined on $[-\lambda, \lambda]$ by

$$h_\lambda(x) = x^2 - \lambda(\lambda - 1)$$

via $g_\lambda = f^{-1} \circ h_\lambda \circ f$, where $f(x) = \lambda(1 - 2x)$. The iterative behavior of g_λ, which has recently intrigued many people (see [7]) is thus completely determined by that of h_λ—and h_λ is rather easier to work with (e.g., see [5]).

Since the days of Babbage, many mathematicians, from Abel [1], through Kuczma [6] to Zimmermann [11], have contributed to the development of iteration theory. Their efforts have been largely, though not exclusively, devoted, first, to generalizing Babbage's problem (1) to the problem of finding, for a given function g and integer n, functions f such that

$$f^n = g, \tag{3}$$

i.e., finding *fractional iterates* of *arbitrary* functions, rather than the identity function alone. Such fractional iterates may or may not exist; and if they do exist, there is generally no uniqueness, not even up to conjugacy.

The fractional iteration equation (3) can in turn be generalized to the problem of defining arbitrary rational, real (or even complex) iterates of a given function g. This leads directly to the notion of an (abstract) *dynamical system*, i.e., a family $\{f_t\}$ of functions such that

$$f_s \circ f_t = f_{s+t} \tag{4}$$

for all s, t in an index set that is closed under addition and may comprise, e.g., the nonnegative integers, the nonnegative rationals, the nonnegative reals, all integers, all rationals, or all reals. If the index set includes nonintegral numbers, and if $f_1 = g$ for a given function g, then the functions f_t can be regarded as generalized iterates of g, and we speak of g as being *embeddable* in such a family of generalized iterates.

Embeddability in a family of generalized iterates, or even the existence of particular fractional iterates depends on the orbit structure of the given function. The notion of *orbit* needed here is that introduced by Kuratowski in a remark at the end of [10]: two points a, b in the union of the domain and range of a function f are in the same f-orbit if there exist nonnegative integers m, n such that $f^m(a) = f^n(b)$. One example, therefore, of an f-orbit, would be a fixed-point of f together with all points that f ultimately sends into that fixed-point. Orbits have natural representations as directed graphs: see [4, 8], and Chapter 1 of the book under review. There is a fundamental connection between the notions of orbit structure and conjugacy: two functions have isomorphic orbit structures if and only if the functions are conjugate.

Orbits, as defined above, have no apparent connection with the ordinary "analytic" properties of functions—measurability, continuity, differentiability, and so on. Indeed, a very well-behaved function, say a real linear function, may be conjugate to, and thus have an orbit structure isomorphic to that of an extremely ill-behaved function, say a nonmeasurable function whose graph is dense in the plane. Nevertheless, connections do exist, the most spectacular one known to date being that contained in the now-celebrated theorem of Sharkovskiĭ [9]: if a continuous function that maps a real interval into itself admits a cycle of order m, then it also admits cycles of all orders n such that $m < n$ in the ordering $<$ defined by

$$3 < 5 < 7 \cdots < 3 \cdot 2 < 5 \cdot 2 < \cdots < 3 \cdot 2^2 < 5 \cdot 2^2 < \cdots < 2^3 < 2^2 < 2 < 1.$$

(That this result remained virtually unknown in the West for over 10 years is evidence of the casual reading habits of many Western mathematicians. A lucid review of [**9**] appeared within a few months of the appearance of the paper itself, in the October 1964 Mathematical Reviews: MR **28** #3121.) Related to this is the "chaotic" behavior exhibited by certain continuous functions under iteration (see [**7**].)

The emergence of such results is one of the two factors that have contributed to the recent upsurge of interest in the iteration of functions. The other factor is the proliferation of computing facilities that make it possible to iterate fairly complicated functions rapidly and display the results, e.g., in the form of scatter plots. Computer experiments have even led some people to the expressed belief that the study of iteration is the key to unlocking the deepest secrets of the universe, or at any rate, the secret of turbulent behavior. Such extravagances are probably temporary; but the associated growth of interest in iteration theory is likely to persist. It is to serve this growth of interest that the book under review was written.

As its title indicates, Professor Targonski's book is selective rather than comprehensive. Attention is focussed primarily on the subjects mentioned above: orbits (and limit sets of orbits), fractional iterates, embeddability of functions in families of generalized iterates, iteration of continuous functions, "chaos". Omitted subjects include iterative methods in numerical analysis, fixed-point theory (apart from the necessary rudiments), and, most regrettably, ergodic theory.

The treatment of the topics considered is reminiscent of that in the old "Encyklopädie" articles: an attempt is made to present all relevant results, with extensive references and cross-references, but without a great deal of discussion. Unlike the Encyklopädie authors, however, Targonski often furnishes proofs. And his discussions, though brief, are consistently illuminating.

Perhaps the most valuable feature of the book is the presentation of material that would otherwise be practically inaccessible. This applies to some of the work of the author himself and his students U. Burkart, R. Graw, and G. Zimmermann. While much of the work of this "Marburg school" has been published, some of it has not, or has only appeared in the form of doctoral dissertations [**2, 3, 11**]. This also applies to the material on the "Pilgerschritt" transformation in Chapter 4. The concept, and the rather odd name, is due to R. Liedl, who introduced it as a method for obtaining (in certain cases) an embedding of a function in a family of generalized iterates by successive approximations. The transformation is most ingenious; its definition is quite involved; and it is good to have a chapter devoted to it. Incidentally, the reason behind the name ("pilgrim's steps" or "pilgrim's walk") is given on p. 126.

The author's writing style is clear and straightforward, but some items of notation may give readers trouble. In particular, the author uses a small Δ in definitions where many people would have $=_{\text{Def.}}$ or, after the words "we set", simply $=$. This is especially confusing in some places where a slightly larger Δ is used to designate something completely different. There is no list of special symbols, so the reader has to stop from time to time and do a bit of deciphering. This is never difficult, but can be annoying. The terms "ω-chain" and "$\omega^* + \omega$-chain" are awkward and could be avoided by having some pictures. Indeed, the addition of pictures, particularly of orbits, would be very helpful. Considerably more attention could have been given to the notion of conjugacy. And one wishes that the book had been set in type, with justified margins, rather than being reproduced directly from typescript.

But these cavils are minor. Professor Targonski has done a great service for all of us interested in iteration theory, and we can thank him by seeing to it that his book sells out as quickly as possible.

References

1. N. H. Abel, *Détermination d'une fonction au moyen d'une équation qui ne contient qu'une seule variable*, Oeuvres, Tome II, pp. 36–39.
2. U. Burkart, *Zur Charakterisierung diskreter dynamischer Systeme*, Inaugural-Dissertation, Phillips-Universität Marburg/Lahn, 1981.
3. R. Graw, *Über die Orbitstruktur stetiger Abbildungen*, Inaugural-Dissertation, Philips-Universität Marburg/Lahn, 1981.
4. R. Isaacs, *Iterates of fractional order*, Canad. J. Math. **2** (1950), 409–416.
5. L. Jonker, *A monotonicity theorem for the family* $f_a(x) = a - x^2$, Proc. Amer. Math. Soc. **85** (1982), 434–436.
6. M. Kuczma, *Functional equations in a single variable*, Monograf. Mat. Tom 46, PWN, Warsaw, 1968.
7. R. M. May, *Simple mathematical models with very complicated dynamics*, Nature **261** (1976), 459–467.
8. R. E. Rice, B. Schweizer, and A. Sklar, *When is* $f(f(z)) = az^2 + bz + c$?, Amer. Math. Monthly **87** (1980), 252–263.
9. A. N. Sharkovskiĭ, *Co-existence of cycles of a continuous mapping of the line into itself*, Ukrain. Mat. Zh. **16** (1964), 61–71. (Russian)
10. R. Tambs Lyche, *Sur l'équation fonctionnelle d'Abel*, Fund. Math. **5** (1924), 331–333.
11. G. Zimmermann, *Über die Existenz iterativer Wurzeln von Abbildungen*, Inaugural-Dissertation, Philipps-Universität Marburg/Lahn, 1978.

Abe Sklar

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 3, November 1983

0273-0979/83 $1.00 + $.25 per page

Birkhoff interpolation, by G. G. Lorentz, K. Jetter and S. D. Riemenschneider, Encyclopedia of Mathematics and its Applications, vol. 19, Addison-Wesley, Reading, Mass., 1982, lv + 237 pp., $32.50. ISBN 0-2011-3518-3

When I learned that G. G. Lorentz was writing a book on Birkhoff interpolation, I was hardly surprised. After all, no one has done more than Lorentz to develop and popularize this topic over the past fifteen years. On the other hand, I had the feeling that perhaps it was premature to commit the subject to book form. For despite considerable progress in understanding the basic problem, the general solution is not in sight and loose ends remain almost everywhere. It was thus with some misgivings that I agreed to write this review. When the copy of the book (coauthored by K. Jetter and S. D. Riemenschneider) arrived, however, I was pleased to find a good deal more