deciphering. This is never difficult, but can be annoying. The terms "ω-chain" and "$\omega^* + \omega$-chain" are awkward and could be avoided by having some pictures. Indeed, the addition of pictures, particularly of orbits, would be very helpful. Considerably more attention could have been given to the notion of conjugacy. And one wishes that the book had been set in type, with justified margins, rather than being reproduced directly from typescript.

But these cavils are minor. Professor Targonski has done a great service for all of us interested in iteration theory, and we can thank him by seeing to it that his book sells out as quickly as possible.

References

1. N. H. Abel, *Détermination d'une fonction au moyen d'une équation qui ne contient qu'une seule variable*, Oeuvres, Tome II, pp. 36–39.

2. U. Burkart, *Zur Charakterisierung diskreter dynamischer Systeme*, Inaugural-Dissertation, Phillips-Universität Marburg/Lahn, 1981.

3. R. Graw, *Über die Orbitstruktur stetiger Abbildungen*, Inaugural-Dissertation, Philips-Universität Marburg/Lahn, 1981.

4. R. Isaacs, *Iterates of fractional order*, Canad. J. Math. **2** (1950), 409–416.

5. L. Jonker, *A monotonicity theorem for the family* $f_a(x) = a - x^2$, Proc. Amer. Math. Soc. **85** (1982), 434–436.

6. M. Kuczma, *Functional equations in a single variable*, Monograf. Mat. Tom 46, PWN, Warsaw, 1968.

7. R. M. May, *Simple mathematical models with very complicated dynamics*, Nature **261** (1976), 459–467.

8. R. E. Rice, B. Schweizer, and A. Sklar, *When is* $f(f(z)) = az^2 + bz + c$?, Amer. Math. Monthly **87** (1980), 252–263.

9. A. N. Sharkovskiĭ, *Co-existence of cycles of a continuous mapping of the line into itself*, Ukrain. Mat. Zh. **16** (1964), 61–71. (Russian)

10. R. Tambs Lyche, *Sur l'équation fonctionnelle d'Abel*, Fund. Math. **5** (1924), 331–333.

11. G. Zimmermann, *Über die Existenz iterativer Wurzeln von Abbildungen*, Inaugural-Dissertation, Philipps-Universität Marburg/Lahn, 1978.

Abe Sklar

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Birkhoff interpolation, by G. G. Lorentz, K. Jetter and S. D. Riemenschneider, Encyclopedia of Mathematics and its Applications, vol. 19, Addison-Wesley, Reading, Mass., 1982, lv + 237 pp., $32.50. ISBN 0-2011-3518-3

When I learned that G. G. Lorentz was writing a book on Birkhoff interpolation, I was hardly surprised. After all, no one has done more than Lorentz to develop and popularize this topic over the past fifteen years. On the other hand, I had the feeling that perhaps it was premature to commit the subject to book form. For despite considerable progress in understanding the basic problem, the general solution is not in sight and loose ends remain almost everywhere. It was thus with some misgivings that I agreed to write this review. When the copy of the book (coauthored by K. Jetter and S. D. Riemenschneider) arrived, however, I was pleased to find a good deal more coherence than I had anticipated. Although there are some shortcomings, I believe that this book will play an important role in spreading awareness of this intriguing, but, heretofore, little known problem, and in illuminating the beautiful mathematics developed in the attempt to solve it. Meticulously documented (over 200 original papers are cited in the bibliography, many unfamiliar even to one who has kept a close watch on the field), the authors succeed in conveying a sense of the current state of the subject to the reader.

Interpolation theory has a long history. As Philip Davis says in his book *Interpolation and approximation* [**5**, p. 24], "This whole book can be regarded as a theme and variation on two theorems: an interpolation theorem of great antiquity and Weierstrass's approximation theorem of 1885. The simple theorem of polynomial interpolation upon which much practical numerical analysis rests says, in effect, that a straight line can be passed through two points, a parabola through three, a cubic through four, and so on". The names of Newton and Lagrange are associated with polynomial interpolation of this type. Later on, the problem of specifying values of a polynomial and *consecutive* derivatives was considered by Hermite [**6**, p. 15]. In both the Lagrange and Hermite cases existence and uniqueness of the interpolant can be established by using elementary techniques of linear algebra and zero counting arguments.

Fascinating results concerning interpolation abound. For example, one might (naively) believe that interpolation is a useful way of obtaining a polynomial approximation to a continuous function (thus providing a proof of Weierstrass's theorem). It is rather startling, therefore, to find that interpolating the function $f(x) = |x|$ at the points k/n, $k = -n, -n+1, \ldots, n-1, n$, yields a sequence of polynomials that diverges everywhere in $[-1, 1]$ except at $x = -1, 0$, and 1 [**6**, p. 39]. Moreover, if $\{x_{nk}\}$, $n = 0, 1, \ldots$; $k = 0, 1, \ldots, n$, is any sequence of interpolation nodes in $[-1, 1]$, then there exists some $f \in C[-1, 1]$ for which the corresponding sequence of interpolants does not converge uniformly [**6**, p. 27]. (This is a consequence of the Uniform Boundedness Principle.) These theorems of negative character are contrasted by the following result of Fejér [**6**, p. 57]: If $f \in C[-1, 1]$ and if $p_{2n-1}(x)$ is defined by $p_{2n-1}(x_{nk}) = f(x_{nk})$, $p'_{2n-1}(x_{nk}) = 0$, where $x_{nk} = \cos((2k-1)\pi/2n)$, $k = 1, 2, \ldots, n$, then $p_{2n-1}(x)$ converges uniformly to $f(x)$ on $[-1, 1]$. *Hermite* interpolation can thus be used to prove the Weierstrass theorem, although Lagrange interpolation cannot.

So what, then, is Birkhoff interpolation? Essentially, it is a development of the past twenty or so years in which many new variations have been played on these old themes. The classical texts on interpolation, such as Steffensen [**9**], Whittaker [**12**], Walsh [**11**], or even the relatively recent book by Cheney [**4**], which has a magnificent section of historical notes, make no mention of Birkhoff interpolation. The first use of the name is by Schoenberg [**8**], who called it Hermite-Birkhoff interpolation. Birkhoff interpolation is a generalization of the Hermite case, obtained by relaxing the requirement of consecutive derivatives at the nodes. When this is done, even the existence of an interpolant becomes questionable. For example, there is no quadratic, $p(x)$, satisfying $p(-1) = p(1) = 0, p'(0) = 1$. On the other hand, there is a unique quadratic satisfying $p'(-1) = y_1, p(0) = y_2, p(1) = y_3$ for any real y_1, y_2, y_3.

The basic problem of Birkhoff interpolation, then, is to determine those configurations of derivatives which always admit unique interpolants. This is, of course, a very general question, and probably cannot be solved in such generality. But a great deal of progress has been made recently, and many special cases have been settled. Of the over 200 references in the bibliography, just 25 predate the aforementioned article of Schoenberg (which appeared in 1966), thus giving an indication of current interest in the field. Although the roots of the problem lie in a 1906 paper of G. D. Birkhoff [**2**] and an important result was established by Pólya [**7**] in 1931, the impetus for the current burst of activity certainly came from Schoenberg. He cast the problem in terms of an incidence matrix: Let $x_1 < x_2 < \cdots < x_k$ be a set of interpolation nodes, and let the matrix $E = (e_{ij})$, $i = 1, 2, \ldots, k$; $j = 0, 1, \ldots, n$, where $e_{ij} = 1$ if $p^{(j)}(x)$ is specified at the node x_i, and $e_{ij} = 0$ otherwise, with exactly $n + 1$ entries equal to 1. E is *poised* (or *regular*) if there is a unique interpolant for *any* choice of the interpolation nodes $\{x_i\}$, $i = 1, 2, \ldots, k$. It is this formulation that has attracted so much attention. Certain classes of incidence matrices have been shown to be poised; many others are now known to be nonpoised. Only the case of 2-row matrices has been completely settled (Pólya [**7**]), and even the 3-row case is quite difficult to analyze. The possibility of obtaining necessary and sufficient conditions for poisedness which can be stated solely in terms of the matrix E seems remote.

Because of the intractability of the basic problem, many offshoots have developed. There are several applications as well; an important one is in the study of uniqueness in constrained approximation [**3**]. Other sets of interpolators have been considered; foremost among these are splines and trigonometric polynomials. Birkhoff quadrature formulas have also been developed. Interpolation at special sets of nodes, such as roots of unity or zeros of certain classical polynomials have been studied. Particularly important here are the results of Turán and his students on lacunary interpolation [**1**, **10**].

All this information and much more will be found in this book. One reservation that I have is that, due to the encyclopedic nature of the work, the essential is occasionally obscured by the peripheral. Additional guidance from the authors would have been useful in certain places. (One should recall, however, that this *is* a volume of an encyclopedia.) This shortcoming is partially offset by a 37-page introduction by Lorentz entitled "Approximation and interpolation in the last 20 years", notes at the end of each chapter, an index, a list of symbols, and the excellent bibliography mentioned earlier. These features provide some direction and help place the subject in a context. Required reading for specialists and recommended to all those interested in discovering modern aspects of a classical problem, this volume is a welcome addition to the mathematical literature.

BIBLIOGRAPHY

1. J. Balazs and P. Turán, *Notes on interpolation*. II, Acta Math. Acad. Sci. Hungar. **8** (1957), 201–215.

2. G. D. Birkhoff, *General mean value and remainder theorems with applications to mechanical differentiation and quadrature*, Trans. Amer. Math. Soc. **7** (1906), 107–136.

3. B. L. Chalmers and G. D. Taylor, *Uniform approximation with constraints*, Jber. Deutsch. Math.-Verein. **81** (1978/79), 49–86.

4. E. W. Cheney, *Introduction to approximation theory*, McGraw-Hill, New York, 1966.
5. P. J. Davis, *Interpolation and approximation*, Blaisdell, New York, 1963.
6. I. P. Natanson, *Constructive function theory*, vol. III, Ungar, New York, 1965.
7. G. Pólya, *Bemerkung zur Interpolation und zur Naherungstheorie der Balkenbiegung*, Z. Angew. Math. Mech. **11** (1931), 445–449.
8. I. J. Schoenberg, *On Hermite-Birkhoff interpolation*, J. Math. Anal. Appl. **16** (1966), 538–543.
9. J. F. Steffensen, *Interpolation*, 2nd ed., Chelsea, New York, 1950.
10. J. Suranyi and P. Turán, *Notes on interpolation*. I, Acta Math. Acad. Sci. Hungar. **6** (1955), 67–80.
11. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, 5th ed., Amer. Math. Soc. Colloq. Publ., no. 20, Amer. Math. Soc., Providence, R.I., 1969.
12. J. M. Whittaker, *Interpolatory function theory*, Stechert-Hafner, New York, 1964.

ELI PASSOW

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The theory of Eisenstein systems, by M. Scott Osborne and Garth Warner, Academic Press, New York, 1981, xiii + 385 pp., $55.00. ISBN 0-1252-9250-3

The Laplace-Beltrami operator on the upper half-plane with respect to the hyperbolic metric is

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

The arithmetic interest of the eigenfunctions of Δ invariant under the modular group $\Gamma = \mathrm{SL}(2, \mathbf{Z})$ and its congruence subgroups was signalled by Maass [**17**], who was inspired by earlier work of Hecke. If $\gamma \in \mathrm{GL}(2, \mathbf{Q})$ then $\Gamma_\gamma = \gamma^{-1}\Gamma\gamma \cap \Gamma$ is of finite index in Γ. Thus if $\det \gamma > 0$ so that γ also acts as a fractional linear transformation on the upper half-plane one can introduce the operator

$$T_\gamma : f \to \sum_{\delta \in \Gamma_\gamma \backslash \Gamma} f(\gamma\delta z), \qquad \mathrm{Im}\, z > 0.$$

It is called a Hecke operator. It commutes with Δ, and acts on its eigenspaces. The study of these operators and of those appearing in Hecke's work promises to be of considerable importance for diophantine problems, in particular for the investigation of the Dirichlet series to which the names of Artin and Hasse-Weil are attached. However the spectral theory of Δ on Γ-invariant functions is a purely analytic problem, of interest in its own right for any discrete subgroup Γ of $\mathrm{SL}(2, \mathbf{R})$ whose fundamental domain has finite volume. If the quotient of the upper half-plane by Γ is compact the spectrum is discrete, but otherwise there is a continuous spectrum and the corresponding eigenfunctions are called Eisenstein series.

If the quotient is not compact there are cusps. By way of illustration we may assume that ∞ is a cusp. This means that Γ contains a subgroup of the form

$$\Gamma_0 = \left\{\begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbf{Z}\right\}$$