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Computational methods for fluid flow, by Roger Peyret and Thomas D. Taylor, Springer Series in Computational Physics, Springer-Verlag, New York, 1983, x + 358 pp., $42.50. ISBN 0-3871-1147-6

I read this book by Peyret and Taylor with pleasure. At last there is an introductory book on computational fluid dynamics that is intelligent, fairly well informed and reasonably up-to-date. Its perspective is not, however, fully consistent with a mathematician's point of view.

Computational fluid dynamics (CFD) is the process of solving problems in fluid dynamics numerically on a computer. It was recognized long ago that the equations of fluid dynamics are particularly amenable to numerical solution, and as early as the 1920's L. Richardson tried to describe how this could be done. In [**18**] he imagined a gigantic concert hall full of hundreds of human "computers" passing pieces of calculations to each other under the majestic guidance of a conductor's baton. The invention of the nonhuman computer greatly streamlined the logistics and since the forties there has arisen an enormous CFD enterprise. There are thousands of researchers, innumerable applications, and an enormous literature; billions of dollars are spent every year on CFD calculations. These calculations have affected an amazing range of sciences—from astrophysics, geophysics, meteorology, biology, and chemistry, to most branches of engineering. Airplanes, space shuttles, weather forecasts, bombs and nuclear power plants are brought to you in part by CFD. Artificial hearts and energy producing devices are being added to this partial list. CFD is useful in all these fields because it supplements or even replaces experiments that are expensive, uninformative or, as in the case of astrophysics and meteorology, uncontrollable.

The equations of fluid dynamics are well known and their validity is not in doubt; one might think that, at least in principle, CFD is merely an elaborate exercise in numerical analysis and approximation theory. However, the phenomena of fluid mechanics (turbulence, combustion, multidimensional shock patterns, boundary layers, etc.) are so complicated that with present methods many of them could not be fully analyzed on any finite computer. As a result, CFD has acquired a life of its own as a major research area—one in which mathematics, physical intuition and computer science interact in original and unexpected ways. Those who are not familiar with fluid mechanics can see

some of the complexity of real flows in the collection of photographs [21], or, if they prefer, they can look at a stormy sky.

Many aspects of CFD are firmly tied to mathematical analysis. The foundations of the subject are numerical analysis and the theory of partial differential equations. Recent developments in CFD use the theory of several complex variables [1], probability theory [4], and various parts of geometry. In return, CFD has contributed enormously to analysis; for example, solitons were first seen in a computer output [22], and the recent numerically assisted work on stochasticity in dynamical systems is an outgrowth of the CFD effort to understand turbulence. Some mathematical and CFD developments go hand in hand; Lax's theories of hyperbolic conservation laws and of differencing in conservation form (see [15, 16]) are parts of a single picture. As was demonstrated by Courant, Friedrichs and Lewy as early as 1928 [9], there is a close relation between numerical methods and constructive existence proofs. A recent example is provided by Glimm's existence proof for nonlinear hyperbolic equations [11], which was loosely suggested by Godunov's computing scheme and has in turn given rise to new algorithms (see [5]); the new algorithms suggested a sharpening of the proof [17], and the sharpened proof led to an improved numerical method [8]. There are many other connections between mathematics and CFD: finite element methods are an aspect of the calculus of variations; a mathematical theory [13, 2, 3] has established the convergence of vortex methods [4, 6] and has generated improvements in the practice of vortex calculations; most of the more innovative practitioners of CFD have been mathematicians, and more mathematicians have contributed to CFD than is commonly realized. Many mathematicians are, for example, surprised by the large fraction of von Neumann's collected works that is connected with CFD.

CFD does, however, have many aspects that do not all conform to the usual definition of mathematics. It uses mathematical constructions well beyond the limits of their rigorously proven validity; algorithms are often validated by internal consistency checks, by comparisons with physical experiments, and by considerations of intrinsic plausibility. A well-chosen example may often be a better guide to the actual performance of a method than a formal proof. Indeed, physical intuition plays a dominant role in the construction of algorithms. Just as there is a notion of mathematical elegance that is difficult to explain to nonmathematicians, there exists a separate notion of "beautiful calculation" that is hard to convey to the uninitiated; roughly speaking, a beautiful calculation is one in which mathematical constructions interact in interesting ways with concrete reality, or one in which a complex physical phenomenon is explained by an unexpected and inexpensive method.

One could think that since CFD has mathematical aspects and is clearly "applied", it is a part of classical applied mathematics. This is not at all the case. CFD's thought patterns are closer to "pure" mathematics—the affinity between algorithms and existence proofs has already been noted. Progress in areas such as turbulence theory occurs through a slow accretion of theorems, calculations and experiments; the kind of analytic or asymptotic relationships that noncomputational applied mathematicians have come to identify with

insight are not always available, and this constitutes a psychological barrier that is often hard to cross. It may be useful to remember that a similar barrier existed in physics when quantum theory was being created. CFD, like quantum theory, requires new, broader and more complex notions of what it means to understand phenomena. Presumably, in the future CFD will come to be seen as an autonomous mathematical science, on the model of statistics or computer science.

Just as the relation between mathematics and CFD has not yet settled into a definitive pattern, the practice of CFD is in a transitional stage and undergoing rapid change. To begin with there are many numerical methods: those based on finite differences, on weak forms of the equations, on variational principles, on expansions in series, on special properties of the equations in specific cases, etc. The relative merits of these methods are the subject of heated discussion. Many methods have devoted adherents who claim that their favorite method solves all problems. The only general rules that can be formulated are general indeed. The right method usually depends on the particular problem and the properties of its expected solution. One should, however, be careful not to force a solution to fit a predetermined pattern. Numerical effects can hide important physical effects; for example, the leading error term in many methods resembles the diffusion term in the equations of motion, and one has to be very careful not to let the former overwhelm the latter. It is not likely that easy consensus will be the hallmark of CFD in the near future. New and unexpected methods are certain to appear, since most of the methods in current use are extensions of methods developed originally for hand computation, and there exist so far only a few methods that utilize powers specific to the computer. Examples are probabilistic methods and Glimm's method, which require, before they produce any useful results, a number of arithmetic operations so large that their use would have been unthinkable in the days of hand computation. Furthermore, it is clear that changing computer architectures will have an effect on algorithms. I believe that we shall also see an increasing use of special purpose methods, such as vortex methods. Vortex methods are based on the circulation theorem for isentropic flow, and cannot be explained without also explaining their hydrodynamical background, in contrast with, for instance, difference methods whose principle can be understood without reference to any specific problem. Special purpose methods can be particularly useful in very complicated problems because they allow one to model the essential properties of specific flows; however, their development is not conducive to the creation of a general theory.

It cannot be too strongly stated that despite its unsettled state, CFD is a very successful enterprise. For example, plane flow at moderate Mach number or moderate viscosity and one-dimensional gas flow can be reliably computed even though interesting mathematical problems remain open. Two-dimensional high-speed gas flow has become accurately computable in recent years, as have some three-dimensional flows, and there have been spectacular successes in transonic flow (see [1] and the bibliography in the book under review). In some areas, such as combusting flow (see e.g. [10]) and vortical flow (see [6, 7])

knowledge is being acquired at a very rapid rate. In many applications, only moderate accuracy is needed for practical purposes, and such accuracy can be provided even when understanding is incomplete.

The situation being as it is, it is not surprising that CFD does not have an established academic home. CFD courses are taught in departments of mathematics, applied mathematics, computer science, chemistry, or one of several engineering departments. Ideally, the computational fluid dynamicist should learn a lot of mathematics (and not only "applied mathematics") as well as a mixture of other sciences. An appropriate curriculum is available at only a few universities. To add to the difficulty of learning the subject, there are very few good CFD books at any level—there are only three or four good specialized monographs and a few sets of notes circulating in a sort of numerical samizdat. (Among the few good monographs one should note [12] on spectral methods, [20] on finite element methods, and [14] on gas dynamics with a Russian flavor.) All in all, it is not easy to become an expert in CFD.

The book under review should be of help. Its most serious flaw is its reluctance to discuss the mathematical background of the various methods, a reluctance born of the desire to appeal to engineering audiences and carried to surprising lengths. The book contains, for example, no proper definition of numerical stability, one of the central concepts in computing. It provides instead a useful, but not infallible, rule for checking whether a given scheme is stable, but one may well wonder what the reader will make of the rule if he does not understand the general principle. The book discusses the projection method for viscous incompressible flow, a very good method for computing such flows that is not often presented at an introductory level, but then the book fails to take advantage of the projection to simplify the discussion of boundary conditions or to analyze the stability of pressure calculations. In fact, the pressure boundary conditions for incompressible flow suggested in the book are both inconsistent and unstable.

Occasionally, the exposition suffers because of the rapid evolution of CFD. For example, the brief discussion of vortex methods is completely out of date; the discussion of Godunov's method, Glimm's method, and, more generally, high resolution methods for hyperbolic systems would have been different if it were being written today (see for example [23]). There are also too many small misstatements and typographical errors.

Despite these defects, this book is a welcome addition to the literature. It presents some of the better methods and makes a laudable effort to compare them with each other. It provides interesting examples, good references, and helpful comments on the literature that should make further reading easier. This is the first good general introduction since the outstanding, but outdated, book of Richtmyer and Morton [19].

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Spectral theory and wave operators for the Schrödinger equation, by A. M. Berthier, Research Notes in Mathematics, Vol. 71, Pitman Advanced Publishing Program, Boston, London, and Melbourne, 1982, 306 pp., $24.95. ISBN 0-2730-8562-X

In the physics literature, many authors discussing quantum phenomena begin by telling us what the classical Hamiltonian (i.e. the generator of the time evolution) for the system would be if the system were classical, and then "quantize" this Hamiltonian. This subjunctive approach to quantum phenomena suggests a facetious operational definition for doing quantum mechanics: taking an attitude towards a physical system which otherwise would behave the way we used to think it behaved. In fact, some of our more glib colleagues never really distinguish whether they are discussing things classically or quantum mechanically. I mention these things to illustrate that even after more than