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Spectral theory and wave operators for the Schrödinger equation, by A. M. Berthier, Research Notes in Mathematics, Vol. 71, Pitman Advanced Publishing Program, Boston, London, and Melbourne, 1982, 306 pp., $24.95. ISBN 0-2730-8562-X

In the physics literature, many authors discussing quantum phenomena begin by telling us what the classical Hamiltonian (i.e. the generator of the time evolution) for the system would be if the system were classical, and then "quantize" this Hamiltonian. This subjunctive approach to quantum phenomena suggests a facetious operational definition for doing quantum mechanics: taking an attitude towards a physical system which otherwise would behave the way we used to think it behaved. In fact, some of our more glib colleagues never really distinguish whether they are discussing things classically or quantum mechanically. I mention these things to illustrate that even after more than

75 years of quantum mechanics, we still seem to need classical mechanics to provide us with a suitable foil for discussing quantum phenomena. At least I do, and I will use it to roughly survey mathematical activities in quantum theory and to locate Berthier's book, *Spectral theory and wave operators for the Schrödinger equation*, among these activities.

This foil of classical mechanics provides a means for identifying (at least) four sorts of mathematical quantum theorists. (1) *Apologists*. These individuals attempt to reconcile quantum with classical mechanics. Bohr and his correspondence principle come to mind, along with current theorists who investigate the semiclassical limit of Planck's constant going to zero (sic). Recently, there has been considerable interest in quantum systems which, if classical, would be ergodic or at least not integrable. (2) *Skeptics*. These individuals search for an as yet undiscovered classical world. Their ranks include hidden variable theorists and, most recently, a school of stochastic mechanicians who identify the evolution given by the Schrödinger equation with a stochastic process. (Their viewpoint can even accommodate spin.) I would assume that the next step for these people is to identify a mechanical "ether" underlying the processes much as the kinetics of liquids or gases underlies Brownian motion. (3) *Formalists*. Their program is to set quantum mechanics on an axiomatic basis or at least a minimal set of principles. Here I would mention von Neumann **[1]**, who translated the physical principles of quantum mechanics, detailed most notably by Dirac **[2]**, into the language of operators in Hilbert space. More recently these researchers have tried to justify the operator-Hilbert space representation of quantum mechanics, starting from a more fundamental (logical) lattice of propositions. From the standpoint of classical mechanics, these individuals are the radicals, ignoring the classical mechanical heritage altogether and leaving the connection with classical mechanics, if any, to the apologists and to the jurists discussed next. (4) *Jurists*. The jurists accept the formalism and tenets of quantum mechanics. But their specific task is to show that the mathematical objects, e.g. operators and measures, which are only formally defined, are in fact well defined and lead to reasonable physics. (Reasonableness might mean, in part, compatibility with classical intuition.) Current examples are the efforts of quantum field theorists to make rigorous sense of the 4-dimensional theories handed to them by the particle theorists, and efforts of mathematicians to prove localization phenomena associated with random potentials. Other jurist activities will be described below. The reader should not infer that my categorization of mathematical quantum theorists is exhaustive or clear cut (in particular, I am confining discussion to the analysis side of matters, omitting the most important algebraic and topological side altogether), nor should he infer that individuals are permitted to work in only one category. It should also be noted that the categories sketched have taken on a life of their own, quite apart from physics.

The book to be reviewed lies in the jurists' category. Let $H = H_0 + V$ be a Schrödinger operator acting in the Hilbert space $L^2(\mathbf{R}^n)$; i.e., $H_0 = -\Delta$, Δ the n-dimensional Laplacian and V is a multiplication operator by a real function $v(x)$ satisfying certain qualitative criteria, e.g., $v(x) \to 0$, $|x| \to \infty$, or belongs to some weighted L^p-space. The goal of the quantum theorist is to "solve" the time-dependent Schrödinger equation

$$i\partial_t\psi = H\psi,$$

$\psi(t) \in L^2(\mathbf{R}^n)$ with $\psi(0) = \psi_0$ a given initial state vector. Clearly, the solution is given by $\psi(t) = \exp(-itH)\psi_0$, with the exponentiated operator defined via the spectral theory for the selfadjoint operator H, but does the solution behave in a reasonable way? In the atomic physics setting, our intuition tells us there are bound states, states which if classical would correspond to the particle(s) being trapped in the vicinity of the nucleus. Following a physically compelling definition for bound state first given by Ruelle, Amrein and Georgescu have shown that these states can be identified with the subspace spanned by the eigenstates of H [3]. Alternatively, one anticipates unbounded motion: if $v(x) \to 0$, $|x| \to \infty$ sufficiently rapidly, and if the particle behaved classically, it could, with enough energy, escape the nucleus and move asymptotically freely. The quantum analogue to this situation is, one might suspect,

$$\exp(-itH)\psi_0 - \exp(-itH_0)\psi_\pm \overset{s}{\to} 0, \qquad t \to \pm\infty,$$

with $\exp(-itH_0)\psi_\pm$ the quantum analogue to free classical motion for some $\psi_\pm$. Indeed, this is the case for a large class of potentials V, although research on this problem continues, particularly for potentials appropriate to the N-body problem. (The transformations $W_\pm : \psi_\pm \mapsto \psi_0$ are known as Møller wave operators. They are of independent mathematical interest since they are isometries intertwining H_0 and continuous parts of H. The physicists' S-matrix describing the collision of particles is constructed from these operators.) Much of Berthier's book is devoted to a discussion of this scattering theory, as it is called, and consists of a readable account of the theories of Kato and Kuroda and of the newer theory of Enss and Mourre.

The book also contains a chapter on Schrödinger operators which have periodic potentials and which serve as simple solid state models. It is a defensible contention that nowhere is quantum theory more striking than in solid state phenomena. Although a classical particle moving in a periodic medium can have any (suitably) positive energy, this is not so quantum mechanically; only certain energy bands are allowable, a fact which lies at the basis of semiconductor and other solid state theory. Although one can well imagine classical states with a particle trapped in one of the "wells" of a periodic potential, there are no quantum bound states; the particle necessarily tunnels through any potential barriers, leaving any bounded region. The author includes the proof that there are no bound states (a problem going back to Titchmarsh) and then proves the immediate corollary that $H = -\Delta + V$, with V not too singular locally, periodic or not, has no eigenfunctions of compact support. This latter fact can be combined with an argument of Kato concerning the asymptotic behavior of eigenfunctions at ∞ to give a proof of nonexistence of positive energy bound states for certain Schrödinger operators, in agreement with our classical intuition that a particle of positive energy escapes the influence of the nucleus and moves off to ∞.

One can imagine books of this sort being written for the physicist who wishes to learn the appropriate functional analysis, or the mathematician who wishes to learn some quantum mechanics [**3**, **4**]. The book serves neither purpose; rather, it seems to be for the mathematician who wishes to study Schrödinger operators for their own sake. But in this regard the book is more introductory, not nearly so penetrating as other works on the subject, for example the books of Simon [**5**, **6**], Glimm and Jaffe [**7**], and especially the highly readable, comprehensive treatises of Reed and Simon [**8**]. The author has been parsimonious with references, particularly in the text, which could frustrate the reader who wishes to pursue the literature further.

Quantum mechanics is a little like its contemporary, Stravinsky's music—very much a part of the standard repertoire and still very revolutionary. The book comes down on the side of repertoire—functional analysis, subheading Schrödinger operators. My guess is that most readers would want a larger perspective, a glimpse of where these operators come from, and why the subject is still provocative.

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Hyperbolic boundary value problems, by Reiko Sakamoto, Cambridge Univ. Press, New York, New York, 1982, viii + 210 pp., $34.50. ISBN 0-5212-3568-5

From its beginning with the study of the vibrations of a stretched string in the eighteenth century, the theory of hyperbolic differential equations has always stood a little apart from the general theory of linear partial differential equations. This was evident in d'Alembert's famous solution formula, in which the wave forms appear explicitly as real function values in such a way as to make natural the ideas of characteristic lines, domains of dependence, and regions of influence. These motifs have carried though the times of Riemann, Goursat, and Hadamard, whose monograph on Cauchy's Problem (the initial