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*Differential operators for partial differential equations and function theoretic applications*, by K. W. Bauer and S. Ruscheweyh, Lecture Notes in Mathematics, Vol. 791, Springer-Verlag, Berlin and New York, 1980, v + 258 pp., $18.00. ISBN 0-3870-9975-1

It has been known for a long time, in a general qualitative way, that solutions of linear partial differential equations of elliptic type, particularly those arising in mathematical physics, share many of the properties of analytic functions of a complex variable. Rendering this phenomenon quantitative and analytic has been a relatively modern development that began in the 1930s with the work of S. Bergman and I. Vekua. Since then it has grown to encompass PDE's of elliptic, parabolic and hyperbolic types into an active field of research referred to as "Function Theoretic Methods in Partial Differential Equations". The general idea is to build a function theory for a specified equation from one that is known a priori. A typical antecedent is analytic function theory, where there is an extensive literature concerning properties such as singularities, analytic continuation, approximation and interpolation, and boundary values.

The development of a function theory for an equation begins by finding an invertible operator that represents a solution as the transform of a unique function called an associate or generator. Analysis of the operator, the associate, and the inverse operator serves to "transplant" the function theory. Because of this transform dependence, it may not be possible to detail specific properties of a solution from those of the associate; for example, singularities from singularities, interpolates from interpolates, and so on. Therefore, alternate representations are sought as effective supplements. These may, in turn, extend the domain to other classes of problems. Integral operator methods have been efficient in function-theoretic studies, and there are relevant books listed as references.

In 1915 G. Darboux introduced differential operators in connection with the Euler equation. But, a function-theoretic approach to PDE's based on transforms that are differential operators did not emerge until the 1960s and to this the authors have contributed significantly. Recently interest in this direction has been increasing because it permits detailed investigations of problems with physical applications. And, subsequently the related theory had grown to the point where a basic reference was needed to outline its development. The book provides this. In addition, one finds that the comprehensive references and inclusion of previously unpublished work support a sense of direction and overview of the "state of the art" for the differential operator method.

The book is focused on three elliptic PDE's and emphasizes the interplay between the associated function theories and physical applications. Transform equivalent problems arise, as well as "miscellaneous" problems which are considered separately. The book is structured as two reports. The first, by Karl Bauer, treats the function theories in a general setting and considers a wide range of applications. The second, by Stephan Ruscheweyh, develops a detailed theory based on analytic functions for one of these equations whose principal part is the second Beltrami operator.

The approach is traditional function-theoretic. The solutions are classical and the associated generators are usually holomorphic or antiholomorphic functions. The analysis is based on representations of solutions to the differential equations

$$(\mathrm{B}_1) \qquad \{\partial_{z\bar z} + A(z,\bar z)\partial_{\bar z} + B(z,\bar z)\}W = 0,$$

$$(\mathrm{B}_2) \quad \{\omega^2\partial_{z\bar z} + (n-m)\varphi'\omega\partial_{\bar z} - n(m+1)\varphi'\bar\psi'\}W = 0, \qquad \omega = \varphi(z) + \bar\psi(z),$$

$$(\mathrm{B}_3) \quad \{(1+\varepsilon z\bar z)^2\partial_{z\bar z} + \varepsilon n(n+1)\}W = 0, \qquad \varepsilon = \pm 1, \quad n, m \in N_0$$

that appear in Bergman canonical form in the complex coordinates $z = x + iy$ and $\bar z = x - iy$, where the Laplacian $\partial_{xx} + \partial_{yy} = 4\partial_{z\bar z}$. The coefficients $A$ and $B$ are analytic on a simply connected domain $D$. Typically, $\varphi$ and $\psi$ are holomorphic or meromorphic in $D$ and share the property (P): (i) have only a finite number of first order poles, (ii) have no common poles, (iii) and have $(\varphi + \bar\psi)\varphi'\bar\psi' \neq 0$.

Although the equations are not really independent, they fall naturally into these forms for the discussion. The first is an extension of the harmonic equation $\partial_{z\bar z}W = 0$. Its solutions arise in the Maxwell system for modelling electric or magnetic $n$-poles, in potential scattering and as the initial data for parabolic PDE's. The next is connected with the Euler-Poisson-Darboux equation which models one-dimensional gas dynamics and the three-dimensional radially symmetric wave equation. The last has the second Beltrami operator for its principal part. Therefore, it is invariant under all rotations of the unit sphere ($\varepsilon = +1$) and all automorphisms of the unit disk ($\varepsilon = -1$) where the principal part is the Laplacian of the respective spherical and hyperbolic geometries.

The first report separates the discussion of $(\mathrm{B}_1)$–$(\mathrm{B}_3)$ into two chapters: representation theory and applications. The chapter on representations incorporates some function-theoretic applications to give perspective. It begins by establishing general conditons for which solutions may be represented as differential operators of order $n$ acting on designated classes of generators. By certain additional requirements on the form of the differential operators, one is led to known representations of the solutions. There are analytic results that flow from the representations, such as "Laurent" series expansions, analytic continuation by the circle-chain method, Riemann functions, and fundamental solutions. A nice illustration follows from the invariance property of $(\mathrm{B}_3)$ which plays a key role in the construction of its automorphic solutions by means of absolute differential invariants. This leads to differential operator representations of all solutions in simply connected domains of the $C$-plane ($\varepsilon = +1$) and of the unit disk ($\varepsilon = -1$).

Once the representations of the solutions are established, they are transformed to other classes of problems by various means. One method is to construct maps by iteration. To sketch a simple example of this, let us consider the solutions of ($B_2$) which are known in differential operator form to include the classical Cauchy-Riemann system, $\partial_{\bar{z}}W = 0$, when $n = 0$. An $m$-fold iteration produces a generalized Cauchy-Riemann system, $(\omega^2\partial_{\bar{z}})^{m+1}W = 0$, and corresponding solutions that for $m = 0$ reduce to the classical problem. A method which is especially effective in extending the differential operator theory is based on the transformation that continues $z \to z_1$ and $\bar{z} \to z_2$ in ($B_1$)–($B_3$) as independent complex variables. Under its action ($B_1$) is formally hyperbolic in $C^2$ and has solutions represented by Bergman and Vekua integral operators. These are converted to integral free (differential operator) form with particular attention paid to solutions whose generators are polynomials of minimal degree. In $C^2$, ($B_3$) is equivalent to an ultrahyperbolic system in $R^4$ so that its solutions may be written as the action of a differential operator on analytic generators. Extensions of the canonical equations to $C^m$ follow. Moreover, parallels which exist between elliptic and parabolic equations are utilized to find simple differential operators that map solutions of the heat equation onto other classes of parabolic equations.

The second chapter begins applications of the representations with a new formula for spherical harmonics. A corresponding class of solutions called hyperboloid functions arises from the wave equation, and a formula for surface harmonics of degree $n$ in $p$ dimensions is treated. A connection is made between the complex potentials of Vekua, $\partial_{z\bar{z}}W - (\partial_z c/c)\partial_{\bar{z}}W - c\bar{c}W = 0$, and the pseudo-analytic functions of L. Bers, $\partial_{\bar{z}}W = aW + b\overline{W}$. The complex potentials can in turn be related to ($B_2$). Because the real and imaginary parts of certain classes of pseudo-analytic functions satisfy PDE's of the type considered here, they have elementary representations on simply connected domains and in the neighborhood of isolated singularities. Depending on the circumstances, pseudo-analytic functions may be described by means of integro-differential operators. And fundamental solutions in the large of the Tricomi equation follow from a combination of assertions about ($B_3$) and analytic continuations.

The differential operator method is linked to more than thirty examples in function theory and mathematical physics in the second chapter. Having noted function-theoretic aspects, we turn to physical applications. An easy way to show their scope and variety is to list some key words and phrases that are found in traditional and modern applications. Topics of the traditional nature are: the iterated Laplacian in $R^2$ and its axially symmetric form in $R^3$, polyharmonic functions, the Tricomi equation (hodograph method for compressible planar fluid flow), and the Riccati equation. Whereas topics falling under the modern applications are: the generalized Stokes-Beltrami system (transonic flows, torsion of shafts, aligned magneto-gas dynamics), linear Backlund transformations (ultrashort optical pulses and long Josephson junctions), and the Ernst equation (a nonlinear PDE arising in general relativity theory in connection with the gravity field of a uniformly rotating source). Thus, the first report includes an interesting collection of important examples.

Karl Bauer's discovery of the differential operator representation for solutions of ($B_3$), which was implicitly known by Darboux, was fundamental to the theory of the first report. In the second report, one finds a detailed analysis of its function-theoretic properties and areas for potential development. The report begins by noting that ($B_3$) is one of the few equations where Vekua's Riemann function is known and introduces Legendre functions of the first and second kinds with appropriate arguments as particular solutions. Later on these become important in applications and in the development of representation theorems for solutions of the Dirichlet problem. The equation is connected with important and thoroughly studied equations such as ones for which certain Eisenstein series are solutions and for which nonautomorphic solutions are studied. And it can be transformed into the generalized axially symmetric potential equation which has been studied by A. Weinstein, R. Gilbert and others.

The second report continues by developing the structure of the solutions on simply connected domains on the Riemann sphere ($\varepsilon = +1$) and the unit disk ($\varepsilon = -1$). This relies on Bauer's representation of the classical solutions as the action of a differential operator on generators that are complex valued harmonic functions. A typical application is geometric, showing that the range of values attained by a solution in $\Omega$ that is continuous on its closure is a subset of the closed convex hull of its restriction to $\partial\Omega$.

For the Dirichlet problem (D) in circles $\Delta$ with continuous boundary values, a unique solution is known for $\varepsilon = -1$. The situation is different if $\varepsilon = +1$ because the coefficient in the equation is positive, and general methods do not apply for existence and uniqueness. An outline of the progress of the development leads to unique solutions of (D) except for finitely many exceptional circles where the boundary values are Hölder continuous. The author completes the solution of (D) by removing these conditions with a particularly simple and nice extension of the Poisson formula. The formula is applied to extend the second Laplace integral for Legendre functions and to obtain a mean-value theorem.

A study of solutions with restricted range is given for the case $\varepsilon = -1$ where a Herglotz type formula develops along with a Harnack inequality for positive solutions on $\Delta$ and a Schwarz lemma. There are several conjectures given concerning the possibility of the existence of Schwarz lemmas by restricting the classes of generators. A Riemann mapping theory is found for univalent subspaces of solutions ($\varepsilon = +1$). This, in particular, shows the power of the function-theoretic method. Certain criteria for univalence are derived when $\varepsilon = -1$. These require the interactions between the geometric theory developed earlier and calculations based on classical function theory using concepts such as subordination.

A theory of Hardy spaces is built that includes a Poisson type representation theorem for solutions in $H^p(\Delta)$ with boundary values in $L^p(\partial\Delta)$ for $p \geqslant 1$. Fatou's theorem and Littlewood's maximal theorem are extended. And, results on "conjugate" solutions in analogy with the M. Riesz and Kolmogoroff theorems follow. The remaining chapters treat summability by the Abel and Cesaro methods, range problems and uniqueness theorems that include discussions of nonisolated zeros and a demonstration of the fact that certain subspaces of solutions are not characterized by their radial limits at the $\partial\Delta$. A big Picard theorem and Schottky's theorem have function-theoretic extensions, as does Hadamard's three circles theorem. Analytic continuations and automorphic solutions are developed at the conclusion.

To summarize, the book is well organized and accurate. Its style is computational. The tables of contents, glossary of terms and references are detailed and timely. And, open questions are pointed out. The result is an informative reference that can be followed with interest.

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*Finite groups and finite geometries*, by T. Tsuzuku (translated from the 1976 Japanese version by A. Sevenster and T. Okuyama), Cambridge Tracts in Mathematics, Vol. 78, Cambridge University Press, 1982, x + 328 pp., $42.50. ISBN 0-5212-2242-7

All finite simple groups are now known.[1] This monumental classification project involved the efforts of numerous mathematicians and occupies many thousands of pages. Several of these group theorists are presently working hard to decrease the size of the proof. Nevertheless, it seems unlikely that the proof of this classification will become accessible to many mathematicians.

How should this classification be viewed by those not in group theory? It is clearly a remarkable result. But is it unapproachable? Is there any point in understanding parts of it? Can it be used outside of group theory, or is it just a marvelous technical feat designed only for internal consumption? It may even be tempting to ask: "What has this done for me lately?"

[1]Except that, as of this writing (April, 1983), the uniqueness of the Monster has not yet been established.