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Optimization—Theory and applications, Problems with ordinary differential equations, by Lamberto Cesari, Applications of Mathematics, vol. 17, Springer-Verlag, New York, 1983, xiv + 542 pp., $68.00. ISBN 0-3879-0676-2

The classical calculus of variations (of functions of one variable) appears to have culminated in the 1940s with Bliss' book [1] and the work of the Chicago school. This classical theory deals with problems typified by the Bolza problem of minimizing an expression of the form

$$g(a, x(a), b, x(b)) + \int_a^b f_0(t, x(t), x'(t))\, dt$$

by a choice of a function $x\colon [a, b] \to \mathbf{R}^n$ that satisfies certain differential equations and boundary conditions. This theory has two basic ingredients, namely necessary conditions and sufficient conditions for minimum, both of an essentially local character.

The classical theory leaves the existence of a minimizing solution an open question. Its necessary conditions may reveal candidates for a local minimum

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and its sufficient conditions may indicate that one or more of these candidates actually yields such a local minimum. However, these conditions may fail to reveal all, or any, of the local minima and they throw no light on the existence of a global minimum. In fact, a global minimum need not exist in general. This is the case even for the simplest problems such as finding a function x: $[a, b] \to \mathbf{R}$ that minimizes $\int_a^b f_0(t, x(t), x'(t))\, dt$, where f_0 may be assumed "nice" (continuous, differentiable, even analytic) and x is chosen from a suitably large class of functions (say, absolutely continuous). Consider, for example, the problem of minimizing

$$I(x) = \int_0^1 \left[x^2 - x'^2 \exp(-x'^2)\right] dt.$$

Since $z^2 \exp(-z^2) \leqslant e^{-1}$ for all real z, it follows that $I(x) \geqslant -e^{-1}$ for all x. This lower bound can never be attained because that would require that $x(t) = 0$ for all $t \in [0, 1]$ and that $x'(t) = 1$ or -1 a.e., two requirments that are incompatible. On the other hand, $-e^{-1}$ is actually the infimum of $I(x)$. Indeed, for $k = 1, 2, \ldots$, let x_k: $[0, 1] \to \mathbf{R}$ be the continuous piecewise linear function vanishing at 0 and with slopes alternately equal to $+1$ and -1 on consecutive intervals of length 2^{-k}. Then $0 \leqslant x_k(t) \leqslant 2^{-k}$, while $|x'(t)| = 1$ a.e. and, therefore, $-e^{-1} < I(x_k) \leqslant 2^{-k} - e^{-1} \to -e^{-1}$.

Aside from leaving unanswered the question of existence, the classical theory was not particularly well suited to handle new problems that arose in practical applications during the period following the second world war. The latter problems, whose relation to the calculus of variations was often overlooked or ignored, appeared in the form of *optimal control problems* that typically require the minimization of an expression such as $g(x(b))$, where x: $[a, b] \to \mathbf{R}^n$ must be absolutely continuous and must satisfy some boundary conditions and the (vector) differential equation $x'(t) = f(t, x(t), u(t))$ a.e. for some choice of the "control function" u with values restricted to some set U (or more generally, $U(t)$ or $U(t, x(t))$). The restriction $u(t) \in U$, usually defined by equalities and inequalities, could have been partly accounted for in the classical theory. (This was done by Valentine [**21**]. In fact, one such problem concerning "the surface of least resistance" was already handled by Newton [**9**, p. 658].) However, the classical theory is an awkward tool for the study of such problems, especially if there are added restrictions such as $x(t) \in A$ for all t, with A a closed set and $x([a, b])$ actually meeting the boundary of A.

The questions of existence for the classical problems of the calculus of variations were to some extent elucidated in the work of Tonelli [**1**] and to a great extent resolved by Young [**29**] and McShane [**11, 12**]. Tonelli proved that if the function $f_0(t, v, \cdot)$ is convex then, subject to certain "growth conditions", there exists a minimizing solution for "free" problems (without associated differential equations). Young introduced weak solutions in the form of "generalized curves" which effectively convexified $f_0(t, v, \cdot)$, and McShane applied generalized curves to the Bolza problem and proved that, subject to certain convexity assumptions, the minimizing generalized curve solution must actually coincide with a conventional solution. (The reader will no doubt notice the similarity to Hilbert's approach to the Dirichlet problem.)

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The first necessary condition and existence theorem for the optimal control of ordinary differential equations were derived in the late 1950s in the U.S.S.R. The necessary condition, now called Pontryagin's maximum principle, represents a generalization of the Weierstrass E-condition, but it is applicable even if the control $\bar{u}(t)$ lies partly on the boundary of U. The first existence theorem was derived by Filippov [7] in 1959 without apparently any knowledge of the work of Tonelli and Young, and it was independently rediscovered, in slightly different forms, by Roxin [**18**], Warga [**22**] and Ważewski [**27, 28**] in 1962 and by Ghouila-Houri [**8**] in 1967. This theorem deals with problems essentially equivalent to the following one: let g: $\mathbf{R}^n \to \mathbf{R}$ be continuous and the sets A, B_0, B_1 and $Q(t, v)$ closed subsets of $\mathbf{R}^n$ for $t \in [a, b]$ and $v \in \mathbf{R}^n$. Consider the class $\mathcal{C}$ of absolutely continuous functions x: $[a, b] \to \mathbf{R}^n$ satisfying the differential inclusion $x'(t) \in Q(t, x(t))$ a.e., the "unilateral" relation $x(t) \in A$ for all t, and the boundary conditions $x(a) \in B_0$, $x(b) \in B_1$. The theorem asserts, in particular, that if $\mathcal{C}$ is nonempty; the sets $Q(t, v)$ are convex and uniformly bounded; the mapping $(t, v) \to Q(t, v)$ is upper semicontinuous with respect to inclusion [i.e. for all (t_0, v_0), any neighborhood of $Q(t_0, v_0)$ contains $Q(t, v)$ for (t, v) close enough to (t_0, v_0)]; and one of the sets A, B_0 or B_1 is bounded, then there exists a function $\bar{x}$ that minimizes $g(x(b))$ over $\mathcal{C}$. This theorem is applicable to optimal control because the differential equation $x'(t) = f(t, x(t), u(t))$ a.e. with $u(t) \in U(t, x(t))$ is equivalent to the differential inclusion

$$x'(t) \in Q(t, x(t)) = f(t, x(t), U(t, x(t))) \quad \text{a.e.}$$

Crudely speaking, Filippov's theorem asserts that if the admissible trajectories x and the sets $Q(t, v)$ are uniformly bounded, then the convexity of $Q(t, v)$ and the upper semicontinuity of $(t, v) \to Q(t, v)$ imply the compactness of the set $\mathcal{C}$ with respect to the sup norm. If $Q(t, v)$ are not convex then we may consider the related *related* problem (referred to by Cesari as *generalized* problem and also known as *sliding regime* or *chattering* problem) in which the original inclusion is relaxed to

$$x'(t) \in \operatorname{co} Q(t, x(t)) \quad \text{a.e.} \qquad (\operatorname{co} = \text{convex hull}).$$

Since elements of $\operatorname{co} Q(t, x(t))$ are convex combinations of those of $Q(t, x(t))$, we can approximate solutions of the relaxed inclusion by using rapidly oscillating "original" controls whose effect is to yield locally averaged rates of change of $x(t)$ approximating elements of $\operatorname{co} Q(t, x(t))$. The relaxed solutions of the optimal control problems are analogous (and closely related) to Young's generalized curve solutions of variational problems.

Some of the work of Lamberto Cesari and of his collaborators since the early 1960s represents the most sustained effort to date to relax many of the assumptions of Filippov's theorem and of related existence theorems and to cast these theorems in a form equally applicable to the calculus of variations and to optimal control. The first and foremost object of attack is the assumption of boundedness of the controls u and of $Q(t, v)$. This assumption is replaced by (weaker) growth conditions whose use goes back to the work of Tonelli around seventy years ago. These growth conditions can be most easily described if we assume (without much loss of generality) that we minimize

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$g(x(b)) = x_1(b)$ (the first component of the vector $x(b)$) and denote by $f_1(t, x(t), u(t))$ the first component of f. Then the growth conditions are typified by the Tonelli-Nagumo condition that assumes the existence of a function ϕ: $[0, \infty] \to \mathbf{R}$, bounded below and such that

$$\lim_{\zeta \to +\infty} \frac{\phi(\zeta)}{\zeta} = +\infty \quad \text{and} \quad f_1(t, v, z) \geqslant \phi(|z|)$$

for $t \in [a, b]$, $v \in A$, $z \in U$. This condition ensures that the optimal control $u(t)$ cannot become very large (in norm) over long periods of time because then x_1' and its integral $x_1(b)$ become large and do not minimize.

A further relaxation of Filippov's assumptions relates to the concept of upper semicontinuity. If the sets $Q(t, v)$ are all uniformly bounded, then the upper semicontinuity of the mapping $(t, v) \to Q(t, v)$ is equivalent to Kuratowski's Property K, namely

$$Q(t_0, v_0) = \bigcap_{\delta > 0} \text{closure} \bigcup_{(t, v) \in N_\delta} Q(t, v) \quad \text{for all } (t_0, v_0),$$

where N_δ is the δ-neighborhood of (t_0, v_0). However, when $Q(t, v)$ are unbounded then property K may hold without upper semicontinuity. Cesari replaces the joint assumption that $(t, v) \to Q(t, v)$ is convex-valued and upper semicontinuous by the weaker "convex" version of Property K, which he names Property Q, and which states that

$$Q(t_0, v_0) = \bigcap_{\delta > 0} \text{closure co} \bigcup_{(t, v) \in N_\delta} Q(t, v) \quad \text{for all } (t_0, v_0).$$

It had been observed in the past (by Tonelli **[20]** and McShane **[10]**) that, in the context of the calculus of variations, existence theorems remained valid in some cases where the growth condition failed to hold on certain exceptional sets. Cesari and his collaborators **[4]** then studied the case of optimal control with no growth assumptions on "slender" sets in the (t, v)-space $\mathbf{R} \times \mathbf{R}^n$. A set S is called *slender* if, for all $i = 1, 2, \ldots, n$ and all sets $\alpha \subset \mathbf{R}$ of (Lebesgue) measure 0, the set

$$S^i(\alpha) = \{v_i \mid (t, v_1, \ldots, v_n) \in S \text{ and } t \in \alpha\}$$

is itself of measure 0.

Cesari's book (to be followed by two other texts, one on parametric problems **[2]** and one on the optimal control of partial differential equations **[3]**) covers most of the above-mentioned topics, together with a number of others that provide the analytic foundations for his arguments or that touch upon peripheral areas. The "old" classical theory of free problems of the calculus of variations is discussed essentially in its conventional form but in a more modern setting based on the Lebesgue integral and absolutely continuous functions x. The necessary conditions for more general variational problems and for problems of optimal control are derived in essentially the same context that was considered by Pontryagin, Boltyanskii, Gamkrelidze and Mishchenko **[16]** around 1961. This leaves out problems with bounded phase coordinates (also known as unilateral problems or problems with restricted state variables) in which the optimal trajectory meets the boundary of the restrictive set

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[**14, 15, 23, 25**]; problems with a variable control set $U(t)$ that are not relaxed [**15, 25**]; problems with nondifferentiable data and specifically with functions $f(t, \cdot, u)$ that are not C^1 [**6, 26**]; and necessary conditions for problems defined by differential inclusions [**5**]. Sufficient conditions are presented in the form derived by Boltyanskii; they are a generalization of the variational conditions based on fields of extremals and on the Hamilton-Jacobi theory.

While the discussion of the necessary and sufficient conditions stops short of many of the developments of the last twenty years, the topic closest to Cesari's interests, namely the existence theory, is covered in great detail which includes many of the most recent results. This subject accounts for about one-half of the book. The study of growth conditions alone takes up Chapters 11–14 and permeates other parts of the book. Chapter 16 is devoted to the existence theorem of L. W. Neustadt and some of its generalizations which apply to the case when

$$f(t, x(t), u(t)) = D(t)x(t) + C(t, u(t)),$$

but no convexity assumptions are made. One pertinent topic that is missing concerns the existence of "discontinuous" optimal trajectories in optimal control, a topic studied by a number of authors in the 1960s [**13, 17, 19, 24**]. (Cesari states in the bibliographical notes on p. 452 that he will discuss such problems elsewhere.)

Cesari lays great stress on the connection of theory to applications and devotes two chapters (3 and 6) to illustrative examples from geometry, mechanics, aerospace science, economics and other fields. In addition, he provides a large number of examples and counterexamples to illustrate the power and the limitations of various theorems. These very welcome features of Cesari's book distinguish it from most other texts on optimal control.

Aside from Young's book [**30**], Cesari's text is perhaps the only one attempting to bridge the gap between the calculus of variations and optimal control theory. This is especially true in existence theory in which the ideas of Tonelli are merged with those of Filippov and of Cesari himself to construct a largely unified framework.

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Statistical analysis of counting processes, by Martin Jacobsen, Lecture Notes in Statistics, Vol. 12, Springer-Verlag, New York, 1982, vii + 226 pp., $14.80. ISBN 0-3879-0769-6

As the implications of a mathematical structure become more deeply understood, the number of applied problems that may be solved by that structure increases rapidly, often in some surprising directions. In *The statistical analysis of counting processes*, Martin Jacobson has given us an excellent account of just

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