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Nonlinear analysis on manifolds. Monge-Ampère equations, by Thierry Aubin, Grundlehren der mathematischen Wissenschaften, vol. 252, Springer-Verlag, Berlin and New York, 1982, xii + 302 pp., $29.50. ISBN 0-3879-0704-1

No one is very surprised if an area of mathematics can solve its own problems. The surprise is when one area of mathematics can help solve those of another. In recent years it has been our good fortune to see problems from places like algebraic geometry and differential topology solved using nonlinear partial differential equations. Of course, an area should not be judged solely on how it helps other branches of mathematics—but the publicity sure helps convince the skeptical of its current relevance. With this in mind, it is important to note that these developments have taken place as part of a vigorous general advance in our understanding of nonlinear partial differential equations.

Linear problems dominated analysis in the first half of this century, which saw the emergence of the now classical linear functional analysis of Hilbert and Banach spaces. A major source of motivation for this work came from an attempt to understand the wave, Laplace, heat, and Schrödinger partial differential equations of mathematical physics. The development of Fourier analysis was one of the most fruitful discoveries, serving simultaneously as both a tool and as a subject in its own right. Now many of the linear differential equations are linear for a very simple reason: one takes a nonlinear equation and bluntly linearizes it (see any derivation of the wave equation

$u_{tt} = c^2 u_{xx}$ for a vibrating string). Although these linear equations are often only an approximation to nonlinear ones, their usefulness is one of the great success stories of mathematics and physics. Hodge theory and the Atiyah-Singer index theorem are two of the better-known purely mathematical recent applications that crucially use linear partial differential equations.

Over the years a collection of nonlinear problems gradually accumulated in the attic. The standard situation was that when a nonlinear differential equation arose, mathematicians made a few tentative jabs, and then departed, leaving a rich legacy to subsequent generations. For nonlinear ordinary differential equations, some real progress was made. The nonlinear pendulum equation, $\ddot{\theta} + k \sin \theta = 0$, and spherical pendulum problem could be solved by the then emerging theory of elliptic functions, while problems in celestial mechanics inspired Poincaré and others to develop "algebraic" topology to help clarify the situation. Riemann also made basic contributions to the study of shock waves in fluid mechanics.

But except for some islands of progress, little was done. Our heritage of unexplored nonlinear problems included such monuments as the many equations from fluid mechanics (such as the Navier-Stokes and Korteweg-deVries equations), elasticity, and differential geometry (including minimal surfaces). A number of these differential equations arise as the Euler-Lagrange equations of a problem in the calculus of variations. The early years of this century added the equations of general relativity and many more nonlinear problems in Riemannian geometry—which was greatly stimulated by its relevance to relativity. Problems accumulated faster than solutions.

Around 1930 the tide began to turn. From the calculus of variations and the geometry of geodesics came Morse theory. At the same time linear partial differential equations had begun to be understood more systematically. Schauder gave a useful generalization of the Brouwer fixed point theorem to Banach spaces, and Leray and Schauder extended the concept of the degree of a map. These tools were specifically developed to apply to nonlinear partial differential equations. In the same years Douglas and Radó made a breakthrough with their existence theorems for minimal surfaces, H. Lewy solved an important case of the Weyl and Minkowski problems in differential geometry, and Leray proved an existence theorem for the Navier-Stokes equations. Progress continued after World War II, including Nash's deep implicit function theorem (see **[H1]**) in 1956.

For the past twenty years it has been clear that the time is ripe to attack nonlinear problems seriously—and with the feeling that there is a reasonable prospect of making headway.

Much of the recent progress on nonlinear problems in geometry boils down to solving some partial differential equation, often of elliptic type. The underlying goal in many such problems is, given a manifold M find an especially "nice object" from which one can more easily read off geometric information, much as one may seek nice coordinates in which a matrix is diagonalized.

The de Rham-Hodge theory is an exemplary model of this procedure. (Here one seeks nice representatives of cohomology classes, and one finds these representatives by solving the linear Laplace equation $\Delta u = f$.) In Riemannian geometry the nice object may be a metric g with constant scalar curvature, or constant Ricci curvature (these are called Einstein metrics), or constant sectional curvature. Part of the investigation in these problems is, of course, determining if there are any obstructions, such as topological ones, to the existence of these objects. The existence of these objects customarily involves solving a partial differential equation.

To someone outside of the subject, these questions may appear narrow and technical. Here are a few applications. Let M satisfy the topological assumptions of the Poincaré conjecture. If one can show that M has a metric with constant sectional curvature, then by known (easy) results one finds that $M = S^n$. In three dimensions it turns out that Einstein metrics (i.e. constant Ricci curvature metrics) are the same as constant sectional curvature metrics. One recent result by R. Hamilton **[H2]** shows that if a three-dimensional manifold has a metric of positive Ricci curvature, then one may deform it to an Einstein metric. While it is not clear if this helps solve the Poincaré conjecture—one would need to find a metric of positive Ricci curvature—it does give very nice metrics for these manifolds; for instance, these Einstein metrics can be used as an alternative to the Meeks-Yau procedure using minimal surfaces in their contribution to the resolution of the Smith conjecture.

Another application is to compact Kähler manifolds (note that this includes all smooth projective algebraic varieties). If the Chern class, c_1, is negative, then one can prove the existence of an Einstein-Kähler metric (Aubin **[Au]**, Yau **[Y]**). Now one knows Chern classes can be written as integrals involving curvature (the simplest case is the Gauss-Bonnet theorem). If one does this computation for the characteristic class $3c_2(M) - c_1(M)^2$ on a surface (complex dimension = 2), then one obtains a complicated expression which H. Guggenheimer **[G]** showed can be written as a sum of squares in the special case of an Einstein-Kähler metric. Combining the last two sentences, one finds Yau's observation **[Y]** that, for a Kähler surface with $c_1 < 0$, $3c_2 - c_1^2 \geqslant 0$; moreover, one can easily see that equality can occur only if M is biholomorphically covered by the ball in $\mathbf{C}^2$.

The above applications all concern constant curvature metrics as the "nice object". In other applications, the nice object is one which minimizes some functional—or at least is a critical point of it. Some useful functionals are: the total scalar curvature for volume-one metrics (gives Einstein metrics), surface area (minimal surfaces), the square of the curvature of a connection (Yang-Mills), and the "energy" of a map between manifolds (harmonic maps, see **[E-L, 1** and **2]**). There have been several recent applications of minimal surfaces by Schoen and Yau **[S-Y]** to positive scalar curvature and to the closely related "positive mass problem" in general relativity; these show how minimal surfaces can be exploited as a tool in much the same way that the corresponding one-dimensional object—geodesics— have been (see also **[K]**). Donaldson **[D]** see also **[F-U]**, has just used Yang-Mills fields, in particular the existence results of Taubes **[T]** and regularity theorem of Uhlenbeck **[U]**, to resolve an important problem in topology. Combined with some other (also recent) purely topological information, one consequence of Donaldson's work is the existence of exotic differentiable structures on $\mathbf{R}^4$ (it had previously been known that $\mathbf{R}^n$, $n \neq 4$, has no exotic structure).

Applications like these should convince even the most skeptical that existence and regularity results in elliptic partial differential equations are powerful tools. The books by Aubin and by Gilbarg and Trudinger **[G-T]** are timely additions to the literature (one should not forget Morrey's book **[M]**, although many find it quite difficult reading). Nonlinear elliptic equations has many technical aspects which may make it appear impenetrable. To begin with, one must face up to working with the Hölder spaces $C^{k+\alpha}$ $(0 < \alpha < 1)$ and the Sobolev spaces H_k^p (derivatives up to order k are in L^p). The novice customarily prefers the spaces C^1, C^2, etc., but these turn out to be unsuitable, except for ordinary differential equations. One way to see this is to consider $\Delta u = f$. If $f \in C^k$ one would hope that $u \in C^{k+2}$, a gain of two derivatives. This is *false*, but it is almost true. In fact, if $f \in C^{k+\alpha}$, then $u \in C^{k+2+\alpha}$; similarly, if $f \in H_k^p$ then $u \in H_{k+2}^p$. Thus, one does gain the desired two derivatives if one works with Hölder and Sobolev spaces. The general theory of elliptic operators is sufficiently well developed that one can often simply read (and understand) the main results that summarize long, difficult, technical results (Aubin's book, and the Appendix in **[B]** contain summaries of many of these technical theorems). Fortunately, it is possible to read these summaries and then immediately launch into the current research literature. At one's leisure one can then go back and fill in the gaps.

For elliptic equations, the basic intuition to be kept in mind is that these equations are very much like equations in $\mathbf{R}^n$. Thus, for the linear equation $Lu = f$ on a compact manifold without boundary, one has the Fredholm alternative: $\operatorname{Im}(L) = (\ker L^*)^{\perp}$, just as for the linear algebraic equation $LX = Y$. In particular, L is surjective if $\ker L^* = 0$. For nonlinear elliptic equations $T(x, u, Du, D^2u) = f$, one uses the same tools and ideas—such as the implicit function theorem and topological methods—that one uses for a system of equations in $\mathbf{R}^n$. To take a specific example, if the linear map L: $\mathbf{R}^n \to \mathbf{R}^n$ is invertible and F: $\mathbf{R}^n \to \mathbf{R}^n$ is continuous and bounded, say $\|F(X)\| \leqslant c$ for all X in $\mathbf{R}^n$, then the equation $LX = F(X)$ has (at least) one solution; one proves this by merely applying the Brouwer fixed point theorem to $X = L^{-1}F(X)$ in the ball $B = \{\|X\| \leqslant R\}$ with $R = c\|L^{-1}\|$. A key step is the *a priori* inequality: if X is a solution then $\|X\| \leqslant R$. The identical assertion and proof apply to the elliptic differential equation $Lu = F(u)$, except that now one uses the Schauder fixed point theorem (Schauder's theorem has an important compactness assumption which is satisfied because the operator L is elliptic). For the differential equation one needs a similar *a priori* inequality on all solutions. In fact, one of the basic lessons of the past is that proving inequalities is usually the main step in solving a nonlinear problem.

As one of the major contributors to the resolution of nonlinear problems in geometry, Thierry Aubin has focused this monograph on a few significant questions on which he has been involved personally, rather than writing a broad treatise. The first half of the book is a survey of required background material from Riemannian geometry, basic analysis, and elliptic partial differential equations. (Some of this "background" does not appear in any other book.) In these earlier chapters—and also occasionally later on in the book—the author often states a result, but refers the reader elsewhere for the proof. Note that some of the material in Chapter 2 on Sobolev spaces is quite recent, especially the sharp values of the constants in the limiting cases of the inequalities. The values of these constants are at the core of some basic problems (see also the recent article of E. Lieb **[L]**, especially his compactness Lemma 2.7).

My own suggestion to the novice is to skim Chapters 1–4 quickly and move to Chapters 5 or 6, where the applications begin. Refer back to earlier chapters to fill in gaps as the need arises, since the specific applications then apply good motivation to appreciate the more technical information, particularly from Chapter 2 on Sobolev spaces.

Chapter 6 on scalar curvature, mainly the Yamabe problem, is one of the highlights of the book. The Yamabe problem seeks a pointwise conformal metric with constant scalar curvature (Yamabe viewed this as a first step in his attempt to solve the Poincaré conjecture using analysis). After a short computation this question reduces to finding a positive solution, $u > 0$, of the deceptively innocent-looking equation

$$\Delta u + c(x)u = \lambda u^{\alpha} \qquad (\lambda = \text{const.}),$$

on a compact n ($\geqslant 3$)-dimensional manifold, with $\alpha = (n+2)/(n-2)$. It is amusing that, if $1 < \alpha < (n+2)/(n-2)$, by using the calculus of variations it is fairly straightforward to solve this equation; the geometry problem (and a related problem for Yang-Mills fields) force one to consider the extreme exponent $\alpha = (n+2)/(n-2)$, where one is at a limiting case of a Sobolev embedding theorem. Here, the function $c(x)$ becomes very important and a delicate analysis is required, in fact, there are many aspects of this problem that are still unresolved, especially in low dimensions (see also the recent articles, **[B-N, L]**). In related work Brezis and Coron **[B-C]** have used similar techniques for the Rellich problem for surfaces of constant mean curvature.

Monge-Ampère equations are discussed in the last two chapters. Chapter 7 concerns a Kähler manifold M. On these the Ricci curvature can be thought of as a closed two-form; consequently it represents a de Rham cohomology class. One can show that this cohomology class is independent of the metric; to be precise, it represents the first Chern class $c_1(M)$ (for real dimension 2 this is the Gauss-Bonnet theorem for surfaces). Two basic questions are:

(i) Does M have an Einstein-Kähler metric?

(ii) If a closed two-form represents $c_1(M)$, then is it the Ricci form of a Kähler metric? (This second question was originally conjectured by Calabi.)

If $g = g_{\alpha\bar{\beta}}$ is a given Kähler metric, then these questions reduce to finding a real function ϕ satisfying the Monge-Ampère equation

$$\det(g + \phi'') = \det(g)e^{f+\lambda\phi},$$

where f is a given function, λ a constant, and ϕ'' is the Hessian,

$$\phi'' = \partial^2\phi/\partial z^{\alpha}\partial\bar{z}^{\beta},$$

with $g + \phi''$ positive definite. This equation was first solved by Aubin [**Au**] for $\lambda > 0$. If the first Chern class is negative, then $g + \phi''$ is the desired Einstein-Kähler metric. Yau [**Y**] reproved this assertion, giving some applications to algebraic geometry; he also gave the first complete proof of the Calabi conjecture by solving the above equation when $\lambda = 0$ (here, the added difficulty is finding an *a priori* uniform bound for the solution). For $\lambda < 0$, in the past few months Futaki [**F**], generalizing the work of Kazdan and Warner [**K-W**], gave an integral condition which is an obstruction to solving the above equation, thus also supplying new examples of Kähler manifolds with first Chern class positive but having no Einstein-Kähler metrics. Another recent development is that Trudinger (see [**G-T**], second edition) has a simplified proof of Evans' local $C^{2+\alpha}$ *a priori* estimate of a solution of a nonlinear uniformly elliptic equation, so that one can now replace the complicated third derivative estimates in this chapter by a more general and conceptually clear argument.

The final chapter considers the Dirichlet problem for real Monge-Ampère equations. Some additional related references are [**C-N-S**] and [**C-Y**].

Aubin's book should make this material much more accessible to those mathematicians who wanted to learn it, but were put-off because the required background and techniques were so widely scattered in the literature.

Because all of the problems in this book lead to elliptic equations, it may be helpful to point out the recent books and survey articles [**C-H**; **H,1**; **N,1** and **2**; **S**; and **W**], which discuss nonlinear problems in differential equations that are not necessarily elliptic.

The subject of nonlinear partial differential equations is still a wilderness. All we know are a few particular examples, and each new example often surprises one with unexpected phenomena. The Korteweg-deVries equation and its amazing ramifications to so many parts of mathematics is one of the most notable instances. There will be more in future years. A tidy systematic understanding still seems to lie in the distant future. But for myself, I think it is more fun when there are so many surprises and rich opportunities. Nonlinear problems are here to stay. To quote Peter Lax, "linearity breeds contempt".

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