We denote by $\mathbb{C}$ the complex plane. If $f$ and $g$ are complex-valued functions on a set $S$, then $C[f,g]$ denotes the algebra of polynomials in $f$ and $g$, with complex coefficients, regarded as functions on $S$.

**Theorem.** Let $1 \leq k \in \mathbb{Z}$, and let $f$ and $g$ be $C^k$ diffeomorphisms of $\mathbb{C}$ into $\mathbb{C}$, having opposite degrees. Then $C[f,g]$ is dense in the Fréchet space $C^k(\mathbb{C})$, i.e., given $h \in C^k(\mathbb{C})$, and $X \subset \mathbb{C}$ compact, there is a sequence $h_n \in C[f,g]$ such that $h_n$ and its derivatives up to order $k$ tend to $h$ and its derivatives, uniformly on $X$.

In case $f(z) = z$ and $g(z) = z$, the Theorem reduces to a result of Weierstrass.

Since each diffeomorphism of the closed unit disc $D$ into $\mathbb{C}$ extends to a diffeomorphism of $\mathbb{C}$ into $\mathbb{C}$, we deduce the following.

**Corollary.** Let $f$ and $g$ be $C^1$ diffeomorphisms of $D$ into $\mathbb{C}$, having opposite degrees. Then $C[f,g]$ is dense in $C(D)$.

This settles an old chestnut in the field of uniform algebras. It remains open whether the Corollary works for $k = 0$, i.e., for all pairs of homeomorphisms of opposite degrees.

**Proof of Theorem.** Without loss of generality, we may take $g = z$, because the chain rule for $D^j(h \circ g)$ is linear in $h$ and involves only $D^j h$ and $D^j g$ for $0 \leq i \leq j$.

Since $f$ has degree $-1$, we deduce that $|f_x| > |f_z|$ on $\mathbb{C}$. In particular, $f_x \neq 0$, so the graph $G = \{(z, f(z)) : z \in \mathbb{C}\}$, which is a $C^k$ submanifold of $\mathbb{C}^2$, has no complex tangents. By the Range-Siu theorem [2], $C^k(G)$ is the closure of the space $\mathcal{O}(G)$ of all functions holomorphic in a neighbourhood of $G$. If we can show that $G$ has an exhaustion by polynomially-convex compact sets, then by the functional calculus [4, Chapter 8], it will follow that $C[z, w]$ is dense in $\mathcal{O}(G)$, and hence in $C^k(G)$; since $z \mapsto (z, f)$ is a $C^k$ diffeomorphism of $\mathbb{C} \to G$, this will imply that $C[z, f]$ is dense in $C^k(\mathbb{C})$. Thus it suffices to show that $X = \{(z, f(z)) : z \in K\}$ is polynomially-convex whenever $K \subset \mathbb{C}$ is a closed disc.

Fix a closed disc $K \subset \mathbb{C}$. By modifying $f$ off $K$, if need be, we may assume $f$ maps $\mathbb{C}$ onto $\mathbb{C}$, that $Df$ and $Df^{-1}$ are bounded and uniformly continuous, and that $|f_x|$ and $1 - |f_x/f_z|$ are bounded away from zero. We need two lemmas, which are essentially classical results of Wermer.

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Lemma 1. There exists a constant \( \lambda_1 > 0 \) such that
\[
(z - a)(f(z) - f(a)) + \lambda f_\xi(a)
\]
is nonzero whenever \( 0 < \lambda < \lambda_1 \), \( a \in \mathbb{C} \), and \( z \in \mathbb{C} \).

Proof. Pick \( \delta > 0 \) such that the modulus of continuity \( \omega(\delta) \) of \( Df \) at \( \delta \) is less than half \( (\inf |f_z|)(1 - \sup_{C} |f_z|) \). Applying the mean value theorem to the real and imaginary parts of \( f \) we deduce that for \( 0 < |z - a| < \delta \), the value \( f(z) - f(a) \) differs from \( f_z(a)(z - a) \) by less than \( 2\omega(\delta)|z - a| \). Thus
\[
\text{Re} \left( \frac{(z - a)(f(z) - f(a))}{f_\xi(a)} \right) > 0
\]
whenever \( |z - a| < \delta \). But for \( |z - a| \geq \delta \),
\[
\left| \frac{(z - a)(f(z) - f(a))}{f_\xi(a)} \right| \geq \frac{\delta^2(\sup |DF^{-1}|)^{-1}}{\inf |f_z|}.
\]
Denoting the right-hand side by \( \lambda_1 \), we see that \( (z - a)(f(z) - f(a))/f_\xi(a) \) omits \( \{z \in \mathbb{C} : 0 < \lambda < \lambda_1\} \), for all \( a \) and \( z \), so the lemma is proved.

Let us denote the uniform closure of \( C[z, f] \) in \( C(K) \) by \( A \).

Lemma 2. Suppose that for each \( a \in K \), there exists a sequence \( \lambda_n \downarrow 0 \) such that \( (z - a)(f(z) - f(a)) + \lambda_n f_\xi(a) \) is invertible in \( A \). Then \( A = C(K) \).

Proof. Briefly, let \( \mu \) be a measure on \( K \), annihilating \( A \). It suffices to show that the Cauchy transform \( \hat{\mu}(a) = \int d\mu(\zeta)/\zeta - a \) vanishes at every point \( a \in K \) at which the Newtonian potential \( \int d\mu(\zeta)/|\zeta - a| \) is finite. But the hypothesis, together with Lemma 1, yields a sequence \( f_n \in A \) such that \( f_n \to (z - a)^{-1} \), pointwise on \( K \sim \{a\} \), and \( |f_n(z)| \leq \text{const} |z - a|^{-1} \). Thus the dominated convergence theorem yields the desired result.

We remark that the hypothesis of Lemma 2 can be weakened to “almost all \( a \in K \)”.

Conclusion of Proof of Theorem. Suppose \( X \) is not polynomially-convex. Then \( A \neq C(K) \), so by Lemma 2, there exists \( a \in K \) and \( \lambda_2 > 0 \) such that for every \( \lambda \) with \( 0 < \lambda < \lambda_2 \), the polynomial \( (z - a)(w - f(a)) + \lambda f_\xi(a) \) has a zero somewhere on the polynomially-convex hull of \( X \). Fix \( \lambda \), with \( 0 < \lambda < \min\{\lambda_1, \lambda_2\} \). Then the family of algebraic curves
\[
(z - a - t)(w - f(a + t)) + \lambda f_\xi(a + t) = 0 \quad (0 \leq t < \infty)
\]
is a curve of algebraic hypersurfaces which meets the hull of \( X \), does not meet \( X \) (by Lemma 1), and goes to the hyperplane at infinity (since \( f \) maps onto \( C \), and \( f_\xi \) is bounded). This contradicts Oka’s characterization of polynomial hulls, as given in [3, (1.2), p. 263]. Thus \( X \) is polynomially-convex, and we are done.

We remark that minor modifications to the foregoing proof permit us to strengthen the Corollary, as follows:

Let \( f \) be an orientation-reversing homeomorphism of \( C \) into \( C \), which is locally \( C^1 \) and noncritical off a closed set \( E \), having area zero and not separating the plane. Then \( C[z, f] \) is dense in \( C(C) \).
Also, for any compact set $X$ in $\mathbb{C}$ and for $0 < \alpha < 1$, suppose $\text{Lip}(\alpha, X)$ denotes the space of bounded functions $g$ of $X$ into $\mathbb{C}$ such that for some $K > 0$, $|g(z) - g(w)| \leq K|z - w|^\alpha$ for all $z, w \in X$ with norm $\sup |g| + \text{Least } K$ and suppose $\text{lip}(\alpha, X)$ denotes those functions $g \in \text{Lip}(\alpha, X)$ such that, given $\epsilon > 0$, there exists $\delta > 0$ such that $|g(z) - g(w)| \leq \epsilon|z - w|^\alpha$ whenever $z$ and $w$ satisfy $|z - w| < \delta$. In view of the results given in [1, p. 227], the conclusion of the above remark implies $\mathbb{C}[z, f]$ is dense in $\text{lip}(\alpha, X)$ for any compact set $X$ in $\mathbb{C}$.

Finally, we remark that the Theorem of this paper is sharp in the sense that one critical point destroys it.

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**DEPARTMENT OF MATHEMATICS, MAYNOOTH COLLEGE, CO. KILDARE, IRELAND**

**DEPARTMENT OF MATHEMATICS, FRAMINGHAM STATE COLLEGE, FRAMINGHAM, MASSACHUSETTS 01701**