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THE HEAT EQUATION AND GEOMETRY OF CR MANIFOLDS[1]

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It is well known that the trace of the heat semigroup for the Laplacian on a compact oriented Riemannian manifold has an asymptotic expansion whose terms are integrals of local geometric invariants; see [**1**, **3**, **4**] and their references. Entirely analogous results are true for the sublaplacian $\Box_b$ on a compact CR manifold. For simplicity, we state results here only for the case of a definite Levi form.

We suppose that the compact CR manifold M has definite Levi form and has been given a Hermitian metric and an orientation; thus there is an inner product in the space $\mathcal{E}^{p,q}$ of forms of type p,q. Let $\mathcal{H}^{p,q}$ be the completion and fix p. The operator

$$\bar\partial_b = \bar\partial_{b,q}\colon \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}$$

has formal adjoint $\mathcal{D}_b$ and gives rise to a nonnegative selfadjoint operator $\Box_b = \Box_{b,q}$ on $\mathcal{H}^{p,q}$ which extends the operator $\mathcal{D}_{b,q}\bar\partial_{b,q} + \bar\partial_{b,q-1}\mathcal{D}_{b,q-1}$. The operator $\Box_{b,q}$ is hypoelliptic for $0 < q < n = \frac{1}{2}(\dim M - 1)$. In the special case that the metric is a Levi metric, there is a canonical metric connection due to Webster [**9**] and C. M. Stanton [**5**].

THEOREM 1. *For $t > 0$ and $0 < q < n$, the operator* $\exp(-t\Box_{b,q})$ *has a smooth kernel $K_{t,q}$. On the diagonal, $K_{t,q}$ has an asymptotic expansion*

$$\text{(1)} \qquad \operatorname{tr} K_{t,q}(x,x) \sim t^{-n-1} \sum_{j=0}^{\infty} t^j K_{j,q}(x)\, dV(x), \qquad t \to 0+,$$

where $\operatorname{tr}\colon \operatorname{Hom} \Lambda^{p,q} \to \Lambda^{2n+1}$ *is the standard map and $dV(x)$ is the volume element. The functions $K_{j,q}$ are locally computable. If the metric is a Levi metric, then $K_{j,q}$ may be computed by evaluating a universal polynomial in the components of the curvature and torsion of the Webster-Stanton connection and their covariant derivatives calculated in normal coordinates.*

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REMARKS. The existence and uniqueness of the kernel $K_{t,q}$, $0 < q < n$, is well known [**6**]. We derive the asymptotic expansions (1) and (2) by constructing a parametrix for $\partial/\partial t + \Box_b$ in a class of pseudodifferential operators which is a suitable modification of that introduced in [**2**] for the purpose of treating $\Box_b$ itself. M. Taylor has obtained somewhat less precise asymptotic expansions using a different pseudodifferential calculus [**8**]. Stanton and Tartakoff [**7**] obtained an exact formula for the kernel $K_{t,q}$, $0 < q < n$, in the case of a Levi metric, using successive approximations to solve an integral equation as in [**4**].

Our pseudodifferential operator calculus shows that the functions $K_{j,q}(x)$ can be computed in terms of the coefficients of certain vector fields, dual 1-forms, and their derivatives. The argument that in the case of a Levi metric these coefficients and derivatives can be expressed in terms of curvature and torsion follows standard lines, as in [**1**] for the torsion zero Riemannian case.

The traces of the matrices $K_{j,t}$, expressed in normal coordinates at x, are $U(n)$-invariants. The use of invariant theory to restrict *a priori* the possible form of the trace as in [**1**, **3**] is complicated here by the presence of torsion. In this respect the situation is similar to the case of Hermitian metrics on complex manifolds.

REFERENCES

1. M. Atiyah, R. Bott and V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279–330; *errata*, ibid. **28** (1975), 277–280.

2. R. Beals and P. C. Greiner, *Pseudodifferential operators associated to hyperplane bundles*, Boll. Sem. Mat. Torino (1983),

3. P. Gilkey, *Curvature and the eigenvalues of the Laplacian for elliptic complexes*, Adv. in Math. **10** (1973), 344–382.

4. H. P. McKean, Jr. and I. M. Singer, *Curvature and eigenvalues of the Laplacian*, J. Differential Geom. **1** (1967), 43–69.

5. C. M. Stanton, *Intrinsic connections for Levi metrics* (in preparation).

6. N. K. Stanton, *The fundamental solution of the heat equation associated with the $\bar\partial$-Neumann problem*, J. Analyse Math. **34** (1978), 265–274.

7. N. K. Stanton and D. S. Tartakoff, *The heat equation for the $\bar\partial_b$-Laplacian* (preprint).

8. M. Taylor, *Noncommutative microlocal analysis*. I (preprint).

9. S. M. Webster, *Pseudo-Hermitian structures on a real hypersurface*, J. Differential Geom. **13** (1978), 25–41.

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