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SOME RESULTS ON BOX SPLINES

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Introduction. The purpose of this note is to describe progress made in understanding the box spline. This function is one of several polyhedral splines intensively studied during the past few years; see [1] for a survey of this subject.

For any set $X = \{x^1, \ldots, x^n\} \subseteq R^s \backslash \{0\}$ with $\langle X \rangle :=$ linear span of $X = R^s$, the box spline is defined by requiring that

$$\int_{R^s} B(x \mid X) f(x)\, dx = \int_0^1 \cdots \int_0^1 f\left(\sum_{i=1}^n t_i x^i\right) dt_1 \cdots dt_n$$

holds for all continuous functions on R^s. $B(x|X)$ is a smooth piecewise polynomial of degree $n-s$ and continuity class $C^{d(X)-1}(R^s)$, where

$$d(X) = \max\{m \colon \langle X \backslash Y \rangle = R^s,\ \forall Y \subset X \ni |Y| = m\}.$$

We are interested in the spline space spanned by integer translates of the box spline

$$S(X) = \langle \{B(\circ - \alpha \mid X) : \alpha \in Z^s\} \rangle.$$

It is important to know for the purpose of approximating smooth functions by scaled translates of box splines what polynomials are in $S(X)$. Denoting by $\Pi(R^s)$ the set of all polynomials on R^s, and by $\mathcal{D}'(R^s)$ the space of Schwartz distributions on $C_0^\infty(R^s)$, it is known that for $X \subset Z^s$,

$$S(X) \cap \Pi(R^s) = D(X),$$

where

$$D(X) = \{f \in \mathcal{D}'(R^s) : D_Y f = 0,\ \forall Y \subset X \ni \langle X \backslash Y \rangle \neq R^s\}$$

and $D_Y = \prod_{y \in Y} D_y$, D_y being the directional derivative in the direction of y.

The Nullstellensatz can be used to show that $\dim D(X) < \infty$ and $D(X) \subset \Pi(R^s)$. This fact and the others mentioned above, as well as relevant references, appear in [1].

Theorems. Our first result is

THEOREM 1. *For any* $X \subset R^s \backslash \{0\}$ *with* $\langle X \rangle = R^s$, $|X| < \infty$, *one has*

$$\dim D(X) = |\mathcal{B}(X)|,$$

where $\mathcal{B}(X) = \{Y : Y \subset X, |Y| = s, \langle Y \rangle = R^s\}$.

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Several special cases of this result can be proved directly, for instance when the vectors in X are in general position or are just multiples of the coordinate vectors. The general case is harder. We employ an induction on s and $|X| = n$ to construct a basis for $D(X)$.

For the next theorem we introduce

$$B(X) = \left\{ x = \sum_{i=1}^{n} t_i x^i : 0 \le t_i \le 1 \right\}$$

and note that $B(X) = \operatorname{supp} B(\circ|X)$.

THEOREM 2.

$$\operatorname{vol}_s B(X) = \sum_{Y \in \mathcal{B}(X)} |\det Y|.$$

SKETCH OF PROOF. For every $Y \in \mathcal{B}(X)$, the set $B(Y)$ is the parallelepiped spanned by Y. One can show that $B(X)$ can be covered by a union of translates of such sets with disjoint interiors. From this fact Theorem 2 is immediate.

In the remainder of our discussion we require that $X \subset Z^s$. The main application of Theorems 1 and 2 is to completely settle the question of local linear independence of translates of box splines. To this end, we will always use Ω for any region on which all the translates of a box spline are polynomials. The local span of box splines relative to Ω is defined by

$$S(X \mid \Omega) = \langle \{ B(x - \alpha \mid X) : \forall \alpha \in Z^s \ni \operatorname{supp} B(\circ{-}\alpha \mid X) \cap \Omega \neq \varnothing \} \rangle.$$

First we count the number of translates whose support intersects Ω (thereby obtaining an upper bound on $\dim S(X \mid \Omega)$).

THEOREM 3. *Let* $X \subset Z^s$, $\langle X \rangle = R^s$ *and define*

$$b(\Omega \mid X) = \{ \alpha : \alpha \in Z^s \ni \operatorname{supp} B(\circ{-}\alpha \mid X) \cap \Omega \neq \varnothing \};$$

then $|b(\Omega \mid X)| = \operatorname{vol}_s B(X)$.

The proof of this result depends on the decomposition of supp $B(\circ|X)$ into translates of the parallelepipeds $B(Y)$, $Y \in \mathcal{B}(X)$, used in Theorem 2, and the observation that any Ω is covered by the same number of translates of parallelepipeds $B(Y)$.

THEOREM 4. (*Local linear independence of box splines*). *Let* $X \subset Z^s$, $\langle X \rangle = R^s$. *Then* $S(X \mid \Omega) = D(X)$. *Furthermore, the following three conditions are equivalent:*

(i) $\sum_\alpha c_\alpha B(x - \alpha \mid X) = 0$, $x \in \Omega$, *implies* $c_\alpha = 0$, $\alpha \in b(\Omega|X)$.

(ii) *For any* $Y \in \mathcal{B}(X)$, *we have* $|\det Y| = 1$.

(iii) $\sum_\alpha c_\alpha B(x - \alpha \mid X) = 0$, $x \in R^s$, *implies* $c_\alpha = 0$, $\alpha \in \mathbf{Z}^s$.

SKETCH OF PROOF. The first assertion follows from the fact that the map $(Tf)(x) = \sum_\alpha f(\alpha) B(x - \alpha \mid X)$ is one-to-one and onto $D(X)$, and the observation that $B(\circ{-}\alpha \mid X)|_\Omega \in D(X)$ (see [**1**]). The equivalence between (ii) and (iii) is known [**1**], while the local linear independence (i) follows from Theorems 1, 2 and 3.

In general, to characterize the linear dependence relations among translates of box splines we need the difference operator analog of $D(X)$ given by

$$\Delta(X) = \{f\colon Z^s \to \mathbf{C} : \Delta_Y f = 0, \forall Y \subset X, \ni \langle X \backslash Y \rangle \neq R^s\}$$

where

$$(\Delta_Y f)(\alpha) = (\prod_{y \in Y} \Delta y) f(\alpha), \qquad (\Delta y f)(\alpha) = f(\alpha + y) - f(\alpha).$$

First we give a complete characterization of $\Delta(X)$.

THEOREM 5. *If* $A(X) = \{z \in \mathbf{C}^s : \exists Y \in \mathcal{B}(X), \ni z^y = 1, y \in Y\}$ *and* $X_z = \{y\colon y \in X, z^y = 1\}$ *then* $f \in \Delta(X)$ *if and only if*

$$f(\alpha) = \sum_{z \in A(X)} z^\alpha p(\alpha \mid z),$$

where $p(x \mid z) \in D(X_z)$ *and, moreover,*

$$\dim \Delta(X) = \mathrm{vol}_s\, B(X).$$

PROOF. Induction on $|X|$ yields $\dim \Delta(X) \leq \mathrm{vol}_s\, B(X)$. On the other hand, one can show that the above sequences are in $\Delta(X)$. Theorems 1 and 2 can then be used to count $\mathrm{vol}_s\, B(X)$ linearly independent such sequences.

REMARK. Since $(1, \ldots, 1) \in A(X)$ and $D(X)$ only contains polynomials, we see by Theorem 5 that $f \in \Delta(X)$ if and only if $f = f_1 + f_2$, where $f_1 \in D(X)$ and f_2 satisfies

$$f_2(\alpha) = \sum_{z \in A(X) \backslash \{(1,\ldots,1)\}} z^\alpha p(\alpha \mid z), \tag{1}$$

where $p(x \mid z) \in D(X_z)$.

Let $E(X)$ denote the subspace of $\Delta(X)$ consisting of all elements of the form (1). Combining the above result with Poisson's summation formula yields

THEOREM 6. *If* $f \in E(X)$ *then*

$$\sum_\alpha f(\alpha) B(x - \alpha \mid X) = 0, \qquad x \in R^s.$$

Next we solve the initial value problem for the family of difference operators Δ_Y, $\langle X \backslash Y \rangle \neq R^s$, which determine $\Delta(X)$.

THEOREM 7. *Any sequence* $\{d_\alpha\colon \alpha \in b(\Omega \mid X)\}$ *has a unique extension* $\{c_\alpha : \alpha \in Z^s\}$ *in* $\Delta(X)$.

SKETCH OF PROOF. From Theorems 3 and 5 we known that $\dim \Delta(X) = |b(\Omega \mid X)|$. One can then use induction on $|X|$ to show that any extension of $d_\alpha = 0$, $\alpha \in b(\Omega \mid X)$, in $\Delta(X)$ must vanish identically.

COROLLARY 1. *Suppose, for any* $Y \in \mathcal{B}(X)$, $|\det Y| = 1$. *Then for any data* y_α, $\alpha \in b(\Omega \mid X)$, *there is a unique* $p \in D(X)$ *such that* $p(\alpha) = y_\alpha$, $\alpha \in b(\Omega \mid X)$.

This result is used to construct a linear projector onto $S(X)$. The construction of this map and details of the proofs will be given elsewhere.

References

1. W. Dahmen and C. A. Micchelli, *Recent progress on multivariate splines*, Approximation Theory IV (C. K. Chui, L. L. Schumaker and J. D. Ward, eds.), Academic Press , New York, 1983, pp. 27–121.

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